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Quantization of Lie–Poisson structures by peripheric chains

Vladimir D Lyakhovsky¹ and Mariano A del Olmo²

¹ Theoretical Department, St Petersburg State University, 198904, St Petersburg, Russia

² Departamento de Física Teórica, Universidad de Valladolid, E-47011, Valladolid, Spain

E-mail: lyakhovs@pobox.spbu.ru and olmo@fta.uva.es

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Abstract

The quantization properties of composite peripheric twists are studied. Peripheric chains of extended twists are constructed for $U(sl(N))$ in order to obtain composite twists with sufficiently large carrier subalgebras. It is proved that the peripheric chains can be enlarged with additional Reshetikhin and Jordanian factors. This provides the possibility to construct new solutions to Drinfeld equations and, thus, to quantize new sets of Lie–Poisson structures. When the Jordanian additional factors are used the carrier algebras of the enlarged peripheric chains are transformed into algebras of motion of the form $\mathfrak{g}_{\mathcal{F}B}^P = \mathfrak{g}_{\mathcal{H}} \ltimes \mathfrak{g}_P$. The factor algebra $\mathfrak{g}_{\mathcal{H}}$ is a direct sum of Borel and contracted Borel subalgebras of lower dimensions. The corresponding ω -form is a coboundary. The enlarged peripheric chains $\mathcal{F}_{\mathcal{F}B}^P$ represent the twists that contain operators external with respect to the Lie–Poisson structure. The properties of new twists are illustrated by quantizing r -matrices for the algebras $U(sl(3))$, $U(sl(4))$ and $U(sl(7))$.

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1. Introduction

Quantum algebras are often considered as deformations of symmetries described by Lie algebraic structures. Many of their physical applications deal with composite systems [1–3]. In these cases the coproducts (i.e. the coalgebraic structure called the co-structure) are relevant [4]. When composite systems (i.e. the sets composed of sub-systems) are studied physicists are well accustomed to work with Lie algebras and with nondeformed Hopf algebras (symmetric or cocommutative coalgebras) since the set of sub-systems is not ordered and for that all measurable results must be independent of the order chosen for the sub-systems (remember the case of the composition of angular momenta).

In general, in a quantum algebra both structures, algebraic and coalgebraic, are deformed. In other words, the coproducts are non cocommutative, thus, quantum algebras are ordered structures. We can interpret this lack of symmetry as an existence of a kind of correlation between the sub-systems. In this way, quantum groups supply us with an approximate symmetry [5].

Twisted quantum algebras have the nice property: one can always conserve the classical Lie algebraic compositions (use the nondeformed basis), only the co-structure is deformed. This is why twisted quantum algebras are interesting objects when physical applications are considered.

It was shown by Drinfeld [6] that a quasitriangular Hopf algebra \mathcal{A} can be transformed into a twisted Hopf algebra when a certain adjoint operator of the ‘twisting’ element of $\mathcal{F} \in \mathcal{A} \otimes \mathcal{A}$ is applied to the coproducts of \mathcal{A} . The algebraic structure of the twisted algebra $\mathcal{A}_{\mathcal{F}}$ is the same as in the original one but the co-structure in $\mathcal{A}_{\mathcal{F}}$ and the quantum \mathcal{R} -matrix are different. Thus, using twists we can obtain quantum deformations of universal enveloping algebras of Lie algebras where the irreducible representations are classical but their compositions (tensor products) are deformed.

Since the seminal paper of Drinfeld mentioned above and especially during the last years important studies were performed in the area of twist deformations and complicated twisting elements defined on large ‘carrier’ algebras were constructed [7–13]. Notwithstanding their complicated form one can always decompose them into the product of basic twisting elements (factors) [14]: Reshetikhin [7], Jordanian [8] or extended [9] factors. The twists are often constructed in the form of multiparametric families and the boundary twisting elements have specific properties. In this concern we can mention the peripheric extended twists which are the limit versions of the extended Jordanian twists [10].

In this work we study the possibility of constructing chains of peripheric extended twists. We shall prove that these chains can be enlarged using the Reshetikhin-like or the Jordanian-like factors. The enlarged chains of twists provide the possibility of deforming universal enveloping Lie algebras in a way different from canonical twisting. The overall adjoint transformation cannot be reduced to the adjoint operators of the carrier algebra of the classical r -matrix. In other words, from a more physical point of view, we can quantize new classical r -matrices, or equivalently, new classical systems described by Poisson–Lie structures [15].

The paper is organized as follows. Section 2 presents a review of the main facts related to the theory of twists that we shall use throughout the paper. In section 3, using the particular case of the Lie algebra $sl(3)$, we demonstrate that the carrier algebra (i.e. the minimal subalgebra of the Lie algebra where the twisting element \mathcal{F} can be defined) associated with a peripheric twist enlarged by a Reshetikhin or Jordanian factor is different from the initial carrier algebra of the peripheric twist. In section 4 we prove the existence of peripheric chains for $sl(N)$, and also find the maximal number of peripheric twisting compositions (or links) that we can add to construct a peripheric chain for $sl(N)$ in the case of even $N = 2n$ or odd $N = 2n - 1$ dimensions. The properties of peripheric chains for $U(sl(N))$ are studied in section 5. We discuss separately two cases depending on the character of the additional twisting factors: Reshetikhin or Jordanian. In the first case the enlarged peripheric chains allow quantization of new classical r -matrices whose associated ω -forms are cohomologically nontrivial. In the second, it is proved that the carrier algebras of the peripheric chains enlarged by the Jordanian factors have the structure of a semidirect sum of Borel and Abelian subalgebras that can be seen as motion algebras. Section 6 presents the enlarged peripheric twists for $U(sl(3))$, $U(sl(4))$ and $U(sl(7))$ in order to illustrate the theory developed in the preceding sections. The discussion of the obtained results and some comments conclude the paper.

2. Mathematical preliminaries

Quasitriangular Hopf algebras $\mathcal{A}(m, \Delta, \eta, \epsilon, S, \mathcal{R})$ can be transformed [6] by an invertible twisting element $\mathcal{F} \in \mathcal{A} \otimes \mathcal{A}$ satisfying the equations

$$\mathcal{F}_{12}(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{23}(id \otimes \Delta)(\mathcal{F}) \quad (\epsilon \otimes id)(\mathcal{F}) = (id \otimes \epsilon)(\mathcal{F}) = 1 \quad (2.1)$$

into twisted Hopf algebras $\mathcal{A}_{\mathcal{F}}(m, \Delta_{\mathcal{F}}, \epsilon, S_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}})$, that have the same multiplication and counit but the twisted coproduct, antipode and \mathcal{R} -matrix defined as follows:

$$\begin{aligned} \Delta_{\mathcal{F}}(X) &= \mathcal{F} \Delta(X) \mathcal{F}^{-1} & S_{\mathcal{F}}(X) &= v S(X) v^{-1} & \mathcal{R}_{\mathcal{F}} &= (\mathcal{F})_{21} \mathcal{R} \mathcal{F}^{-1} \\ \mathcal{F} &= \sum f_i^{(1)} \otimes f_i^{(2)} & X &\in \mathcal{A} & v &= \sum f_i^{(1)} S(f_i^{(2)}) \end{aligned} \quad (2.2)$$

Twists are mostly used as tools to construct quantum deformations, $\mathcal{F} : U(\mathfrak{g}) \rightarrow U_{\mathcal{F}}(\mathfrak{g})$, of universal enveloping algebras $U(\mathfrak{g})$ of simple Lie algebras \mathfrak{g} , considered as Hopf algebras with primitive generators $L \in \mathfrak{g}$ (i.e. $\Delta(L) = L \otimes 1 + 1 \otimes L$). The minimal subalgebra $\mathfrak{g}_c \subset \mathfrak{g}$ on which the twist \mathcal{F} can be defined is called the carrier of the twist \mathcal{F} .

In general, the composition of twists is not a twist. But there are some important examples of the opposite behaviour. In particular, a composition of twists \mathcal{F}_1 and \mathcal{F}_2 (with carriers \mathfrak{g}_{c_1} and \mathfrak{g}_{c_2}) can be performed when in the twisted algebra $U_{\mathcal{F}_1}(\mathfrak{g})$ one can find the primitive generators for the carrier \mathfrak{g}_{c_2} .

There are three basic twisting factors (BTFs) [14]. Up to now all the twists that are known in the explicit form are composed of these BTFs:

1. The Reshetikhin basic twisting factor (RF) is defined for any pair of commuting primitive elements l_i, l_j in \mathcal{A} [7] by

$$\Phi_R = e^{\psi_{i,j} l_i \otimes l_j}. \quad (2.3)$$

The coefficient $\psi_{i,j}$ plays the role of the deformation parameter.

2. The Jordanian BTF (JF) [8] has as a carrier the two-dimensional (2D) Borel algebra \mathbf{B}

$$[H, E] = E. \quad (2.4)$$

For $U(\mathbf{B})$ the following element is the solution of (2.1):

$$\Phi_J = e^{H \otimes \sigma} \quad (2.5)$$

with

$$\sigma = \ln(1 + E). \quad (2.6)$$

The result of the deformation $\Phi_J : U(\mathbf{B}) \rightarrow U_J(\mathbf{B})$ is a Hopf algebra $U_J(\mathbf{B})$ with the coproducts:

$$\Delta_J(H) = H \otimes e^{-\sigma} + 1 \otimes H \quad \Delta_J(E) = E \otimes e^{\sigma} + 1 \otimes E. \quad (2.7)$$

The deformation parameter is introduced by scaling the element $E \in \mathbf{B}$:

$$E \longrightarrow \xi E \quad \sigma \longrightarrow \sigma(\xi) = \ln(1 + \xi E) \quad \Phi_J \longrightarrow \Phi_J(\xi) = e^{H \otimes \sigma(\xi)}. \quad (2.8)$$

3. The third BTF [9], the extending factor (EF) or simply the extension, is defined for the deformed universal enveloping algebra $U_J(\mathbf{H})$, where \mathbf{H} is the 3D Heisenberg algebra

$$[A, B] = E \quad [A, E] = 0 \quad [B, E] = 0 \quad (2.9)$$

with the coproducts

$$\begin{aligned} \Delta_J(A) &= A \otimes e^{\alpha\sigma} + 1 \otimes A \\ \Delta_J(B) &= B \otimes e^{\beta\sigma} + 1 \otimes B \\ \Delta_J(E) &= E \otimes e^{\sigma} + 1 \otimes E \end{aligned} \quad \alpha + \beta = 1 \quad (2.10)$$

and σ as in (2.6). The extending factors

$$\Phi_E = e^{A \otimes B e^{-\beta\sigma}} \quad \Phi'_E = e^{-B \otimes A e^{-\alpha\sigma}} \quad (2.11)$$

are the solutions of the twist equations (2.1) and induce the deformations $\Phi_E : U_J(\mathbf{H}) \rightarrow U_E(\mathbf{H})$ or $\Phi'_E : U_J(\mathbf{H}) \rightarrow U_{E'}(\mathbf{H})$. For example, the co-structure of the algebra $U_E(\mathbf{H})$ is defined by the coproducts:

$$\begin{aligned} \Delta_E(A) &= A \otimes e^{-\beta\sigma} + 1 \otimes A \\ \Delta_E(B) &= B \otimes e^{\beta\sigma} + e^\sigma \otimes B \\ \Delta_E(E) &= E \otimes e^\sigma + 1 \otimes E. \end{aligned} \quad (2.12)$$

EFs were first considered as parts of the so-called extended Jordanian twists (EJs) [9]. In contrast to the other two BTFs these are the discrete solutions of the twist equations, though they can borrow a continuous parameter from a smooth curve of equivalent algebras $U_J(\mathbf{B}; \sigma(\xi))$, where $\sigma(\xi)$ is the same as in the parametrized Jordanian twist (see (2.8)).

Note that usually the EF is applied to the universal enveloping algebras $U_J(\mathbf{L}(\alpha, \beta))$ with the 4D carrier subalgebra $\mathbf{L}(\alpha, \beta)$. The parameters α and β that appear in (2.10) are to coincide with the corresponding eigenvalues of the operator $\text{ad}(H)$:

$$\begin{aligned} [H, E] &= E & [H, A] &= \alpha A & [H, B] &= \beta B \\ [A, B] &= E & [E, A] &= 0 & [E, B] &= 0 & \alpha + \beta &= 1. \end{aligned} \quad (2.13)$$

This Hopf algebra is already deformed by the Jordanian factor. The two factors are then considered together forming the extended Jordanian twist [9], $\mathcal{F}_{EJ} = \Phi_E \Phi_J$. This twist defines the deformed Hopf algebras $U_{\mathcal{E}}(\mathbf{L}(\alpha, \beta))$ with the co-structure

$$\begin{aligned} \Delta_{\mathcal{E}}(H) &= H \otimes e^{-\sigma} + 1 \otimes H - A \otimes B e^{-(\beta+1)\sigma} \\ \Delta_{\mathcal{E}}(A) &= A \otimes e^{-\beta\sigma} + 1 \otimes A \\ \Delta_{\mathcal{E}}(B) &= B \otimes e^{\beta\sigma} + e^\sigma \otimes B \\ \Delta_{\mathcal{E}}(E) &= E \otimes e^\sigma + 1 \otimes E. \end{aligned} \quad (2.14)$$

Also we would like to recall that any BTF can be deformed by the previous factors of the sequence of twists. Nevertheless, it always has the form of an exponent whose argument is a tensor product of two elements of \mathcal{A} . Moreover, when in the sequence of twists all the parameters independent of the proper parameter of the BTF are equal to zero the BTF recovers its canonical form.

In [10] the extended Jordanian twists were studied in the limit points $\alpha = 0$ (or $\beta = 0$). The corresponding twists were called peripheric extended twists (PETs). It is easy to see from (2.12) that in these limiting points some generators of $\mathbf{L}^{\mathcal{P}} = \mathbf{L}(1, 0)$ remain primitive in the deformed $U_{\mathcal{P}}(\mathbf{H})$. For example, in $U_{\mathcal{P}}(\mathbf{L}^{\mathcal{P}})$ we have

$$\begin{aligned} \Delta_{\mathcal{E}}^{\mathcal{P}}(H) &= H \otimes e^{-\sigma} + 1 \otimes H - A \otimes B e^{-\sigma} \\ \Delta_{\mathcal{E}}^{\mathcal{P}}(A) &= A \otimes 1 + 1 \otimes A \\ \Delta_{\mathcal{E}}^{\mathcal{P}}(B) &= B \otimes 1 + e^\sigma \otimes B \\ \Delta_{\mathcal{E}}^{\mathcal{P}}(E) &= E \otimes e^\sigma + 1 \otimes E. \end{aligned} \quad (2.15)$$

The second cohomology group $H^2(\mathbf{L}^{\mathcal{P}})$ of the carrier algebra for the PET is nontrivial. We have $\dim H^2(\mathbf{L}^{\mathcal{P}}) = 1$ (in contrast to the case $\alpha, \beta \neq 0, 1$). The nontrivial cocycle can be chosen to be proportional to $H^* \wedge A^*$. This means that $\mathbf{L}^{\mathcal{P}}$ has not only the nondegenerate

coboundary $\omega = E^*([,])$ but also the nondegenerate cocycle [16]

$$\omega_{\mathcal{RE}} = E^*([,]) + \xi H^* \wedge A^*. \quad (2.16)$$

The latter defines the classical r -matrix $r_{\mathcal{RE}} = H \wedge E + A \wedge B + \psi A \wedge E$. It is easy to verify that the corresponding twisting element is a composition of the PET $\mathcal{F}_{\mathcal{E}}^{\mathcal{P}}$ and the RF

$$\Phi_R = e^{\psi A \otimes \sigma} \quad (2.17)$$

i.e.

$$\mathcal{F}_{\mathcal{RE}}^{\mathcal{P}} = \Phi_R \mathcal{F}_{\mathcal{E}}^{\mathcal{P}} = e^{\psi A \otimes \sigma} e^{A \otimes B} e^{H \otimes \sigma}. \quad (2.18)$$

Other specific possibilities of PETs arise when $\mathbf{H}(\beta = 0)$ is considered as a subalgebra of a simple Lie algebra \mathfrak{g} . The most interesting are the non-Abelian primitive subalgebras in \mathfrak{g} that contain the generators such as A (see section 4 where these possibilities are studied in detail). In the simplest case when the initial twist is a peripheric twist with several extension factors $\mathcal{F}_{\mathcal{E}}^{\mathcal{P}}$ these additional possibilities lead to the deformations equivalent to the multi-Jordanian twistings (see section 6).

More fruitful and much more complicated is the situation where the initial twist is a nontrivial composition of extended twists. For Hopf algebras $U(\mathfrak{g})$ with classical simple Lie algebras \mathfrak{g} there exists possibility of constructing systematically the compositions of twists called chains [11]:

$$\mathcal{F}_{\mathcal{B}_{p=0}} \equiv \mathcal{F}_{\mathcal{B}_p} \mathcal{F}_{\mathcal{B}_{p-1}} \dots \mathcal{F}_{\mathcal{B}_0}. \quad (2.19)$$

The factors $\mathcal{F}_{\mathcal{B}_k} = \Phi_{\mathcal{E}_k} \Phi_{\mathcal{J}_k}$ of the canonical chain are the canonical extended Jordanian twists for the initial Hopf algebra \mathcal{A} . Their carriers are the multidimensional analogues to $\mathbf{L}(1/2, 1/2)$, the extensions $\{\Phi_{\mathcal{E}_k}, k = 0, \dots, p-1\}$ contain the fixed set of normalized factors such as $\Phi_{\mathcal{E}} = \exp\{A \otimes B e^{-\frac{1}{2}\sigma}\}$. The sequence of twists in a chain for the Lie algebra $\mathcal{A} = U(\mathfrak{sl}(N))$ corresponds to the sequence of injections $U(\mathfrak{sl}(N)) \supset U(\mathfrak{sl}(N-2)) \supset \dots \supset U(\mathfrak{sl}(N-2k)) \dots$. For each \mathcal{A}_k the initial root λ_0^k is fixed. The extensions $\Phi_{\mathcal{E}_k}$ are defined by the set π_k of constituent roots $\pi_k = \{\lambda', \lambda'' \mid \lambda' + \lambda'' = \lambda_0^k; \lambda' + \lambda_0^k, \lambda'' + \lambda_0^k \notin \Lambda_{\mathcal{A}}\}$, where $\Lambda_{\mathcal{A}}$ is the root system of \mathcal{A}_0 . The set of constituent roots can be naturally decomposed as $\pi_k = \pi'_k \cup \pi''_k$, $\pi'_k = \{\lambda'\}$, $\pi''_k = \{\lambda''\}$. In these terms the factors $\Phi_{\mathcal{E}_k}$ and $\Phi_{\mathcal{J}_k}$ have the form

$$\Phi_{\mathcal{J}_k} = \exp\{H_{\lambda_0^k} \otimes \sigma_0^k\} \quad \sigma_0^k = \ln(1 + L_{\lambda_0^k}) \quad (2.20)$$

$$\Phi_{\mathcal{E}_k} = \prod_{\lambda' \in \pi'_k} \Phi_{\mathcal{E}_{\lambda'}} = \prod_{\lambda' \in \pi'_k} \exp\{L_{\lambda'} \otimes L_{\lambda_0^k - \lambda'} e^{-\frac{1}{2}\sigma_0^k}\} \quad (2.21)$$

where the generator L_{λ} is associated with the root λ .

The role of additional twistings provided by the chains of twists with the peripheric properties and the new class of Lie–Poisson structures that can be thus quantized are studied in section 5.

3. Deformations of carriers: the $\mathfrak{sl}(3)$ case

Most interesting are the cases where the additional factors change the carrier subalgebra. This may happen if the initial carrier $\mathbf{L}^{\mathcal{P}}$ is a proper subalgebra $\mathbf{L}^{\mathcal{P}} \subset \mathfrak{g}$ and the Drinfeld equations are considered over \mathfrak{g} . In this section the simplest variant of such situation is studied. Suppose that \mathfrak{g} contains the Lie subalgebra $\mathbf{M} = H^{\perp} \triangleright \mathbf{L}^{\mathcal{P}}$, which is the extension of $\mathbf{L}^{\mathcal{P}}$ by the 1D subalgebra generated by H^{\perp} with the following action on $\mathbf{L}^{\mathcal{P}}$:

$$[H^{\perp}, E] = 0 \quad [H^{\perp}, A] = A \quad [H^{\perp}, B] = -B \quad [H^{\perp}, H^{\mathcal{P}}] = 0.$$

When $\mathfrak{g} = \mathfrak{sl}(3)$ the generators of $\mathbf{M} \subset \mathfrak{sl}(3)$ can be identified as follows:

$$\begin{aligned} H^{\mathcal{P}} &= \frac{1}{3}(2E_{11} - E_{22} - E_{33}) & A &= E_{12} & E &= E_{13} \\ H^{\perp} &= \frac{1}{3}(E_{11} - 2E_{22} + E_{33}) & B &= E_{23}. \end{aligned} \quad (3.1)$$

In $U_{\mathcal{P}}(\mathbf{M})$ twisted by $\mathcal{F}_{\mathcal{E}}^{\mathcal{P}}$ the co-structure is defined by (2.15) while the coproduct of the additional generator H^{\perp} remains primitive, $\Delta_{\mathcal{P}}(H^{\perp}) = H^{\perp} \otimes 1 + 1 \otimes H^{\perp}$. If we want to change the carrier $\mathbf{L}^{\mathcal{P}}$ of the twist by introducing an extra factor (to enlarge the $\mathcal{F}_{\mathcal{E}}^{\mathcal{P}}$) we have to consider the primitive Borel subalgebra $\mathbf{B} \subset \mathbf{M}$ generated by $\{A, H^{\perp}\}$ or the primitive Abelian subalgebra \mathbf{A} generated by $\{\sigma, H^{\perp}\}$. This means that $U_{\mathcal{P}}(\mathbf{M})$ admits additional Jordanian or Reshetikhin twists

$$\Phi_{\mathcal{J}}^{\perp} = \exp\{H^{\perp} \otimes \sigma_A(\xi)\} \quad \text{or} \quad \Phi_{\mathcal{R}} = \exp\{\zeta H^{\perp} \otimes \sigma\}.$$

Composing $\Phi_{\mathcal{J}}^{\perp}$ or $\Phi_{\mathcal{R}}$ with $\mathcal{F}_{\mathcal{E}}^{\mathcal{P}}$ we obtain new solutions of the Drinfeld equations for the algebra $U(\mathfrak{sl}(3))$:

$$\mathcal{F}_{\mathcal{J}\mathcal{E}}^{\mathcal{P}} = \Phi_{\mathcal{J}}^{\perp} \mathcal{F}_{\mathcal{E}}^{\mathcal{P}} = e^{H^{\perp} \otimes \sigma_A(\xi)} e^{A \otimes B} e^{H^{\mathcal{P}} \otimes \sigma} \quad \mathcal{F}_{\mathcal{J}\mathcal{E}}^{\mathcal{P}} : U(\mathbf{M}) \rightarrow U_{\mathcal{P}}(\mathbf{M}) \rightarrow U_{\mathcal{J}\mathcal{P}}(\mathbf{M}) \quad (3.2)$$

and

$$\mathcal{F}_{\mathcal{R}\mathcal{E}} = \Phi_{\mathcal{R}} \mathcal{F}_{\mathcal{E}}^{\mathcal{P}} = e^{\zeta H^{\perp} \otimes \sigma} e^{A \otimes B} e^{H^{\mathcal{P}} \otimes \sigma} \quad \mathcal{F}_{\mathcal{R}\mathcal{E}} : U(\mathbf{M}) \rightarrow U_{\mathcal{E}}(\mathbf{M}) \rightarrow U_{\mathcal{R}\mathcal{E}}(\mathbf{M}).$$

The universal elements

$$\mathcal{R}_{\mathcal{J}\mathcal{E}} = (\mathcal{F}_{\mathcal{J}\mathcal{E}}^{\mathcal{P}})_{21} (\mathcal{F}_{\mathcal{J}\mathcal{E}}^{\mathcal{P}})^{-1} \quad \mathcal{R}_{\mathcal{R}\mathcal{E}} = (\mathcal{F}_{\mathcal{R}\mathcal{E}})_{21} (\mathcal{F}_{\mathcal{R}\mathcal{E}})^{-1}$$

are the quantizations of the classical r -matrices

$$r_{\mathcal{J}\mathcal{E}} = H^{\mathcal{P}} \wedge E + A \wedge B + \xi H^{\perp} \wedge A \quad r_{\mathcal{R}\mathcal{E}} = H^{\mathcal{P}} \wedge E + A \wedge B + \zeta H^{\perp} \wedge E. \quad (3.3)$$

They define the dual Lie algebras $\mathbf{M}_{\mathcal{J}^{\perp}}^*$, $\mathbf{M}_{\mathcal{R}}^*$ and the Lie algebra morphisms $\mathbf{M}^* \rightarrow \mathbf{M}$. Applying them to $\mathbf{M}_{\mathcal{J}^{\perp}}^*$ and $\mathbf{M}_{\mathcal{R}}^*$ we get (as an image of this map and after the appropriate redefinition of the generators) the 4D subalgebras $\mathbf{L}_{\mathcal{J}^{\perp}}$, $\mathbf{L}_{\mathcal{R}} \subset \mathbf{M}$ with commutators

$$\begin{aligned} \mathbf{L}_{\mathcal{J}^{\perp}} &\begin{cases} [H, E] = E & [H, A] = A & [H, B] = 0 \\ [A, B] = E & [E, A] = 0 & [E, B] = \xi E \end{cases} \\ \mathbf{L}_{\mathcal{R}} &\begin{cases} [H, E] = E & [H, A] = (1 - \zeta) A & [H, B] = \zeta B \\ [A, B] = E & [E, A] = 0 & [E, B] = 0. \end{cases} \end{aligned}$$

The algebras $\mathbf{L}_{\mathcal{J}^{\perp}}$, $\mathbf{L}_{\mathcal{R}}$ and $\mathbf{L}^{\mathcal{P}}$ are inequivalent. The deforming functions $\mu_{\mathcal{J}^{\perp}}(E, B) = E$ and $\{\mu_{\mathcal{R}}(H, B) = B, \mu_{\mathcal{R}}(H, A) = -A\}$ are cohomologically nontrivial: $\mu_{\mathcal{J}^{\perp}}, \mu_{\mathcal{R}} \in H^2(\mathbf{L}^{\mathcal{P}}, \mathbf{L}^{\mathcal{P}})$. Here it is worth mentioning that both deforming functions describe not only the ‘tangent vector’ to the deformation curve but the complete (first order) deformation of \mathbf{M} . Note that $\mathbf{L}_{\mathcal{J}^{\perp}}$ and $\mathbf{L}_{\mathcal{R}}$ are Frobenius Lie algebras. The corresponding nondegenerate forms are coboundaries.

In this simple example we have demonstrated that the carrier subalgebra associated with the peripheric twist $\mathcal{F}_{\mathcal{E}}^{\mathcal{P}}$ enlarged by $\Phi_{\mathcal{J}}^{\perp}$ or $\Phi_{\mathcal{R}}$ differs nontrivially from the initial $\mathbf{L}^{\mathcal{P}}$.

4. Construction of peripheric chains

The constructions used in the previous section can be generalized for the case where the initial carrier algebra contains the maximal nilpotent subalgebra of a simple Lie algebra. The only known twists with such properties are the (full) chains of extended Jordanian twists (2.19). In order to use chains of twists for our purposes we must find their analogues with the peripheric properties. In [17] it was proved that the peripheric chains exist.

In this section we shall compose and study the full peripheric chains for $\mathfrak{g} = \mathfrak{sl}(N)$. We normalize the Cartan elements as $H_{i,k} = (E_{ii} - E_{kk})/2$ and use the standard $\mathfrak{gl}(N)$ -basis $\{E_{i,j}\}_{i,j=1,\dots,N}$. Consider the canonical chain (2.19) of extended twists $\mathcal{F}_{\mathcal{B}_{p<0}} \equiv \mathcal{F}_{\mathcal{B}_p} \mathcal{F}_{\mathcal{B}_{p-1}} \dots \mathcal{F}_{\mathcal{B}_0}$ with

$$\mathcal{F}_{\mathcal{B}_k} = \Phi_{\mathcal{E}_k} \Phi_{\mathcal{J}_k} = \left(\prod_{s=k+2}^{N-k-1} e^{E_{k+1,s} \otimes E_{s,N-k}} e^{-\frac{1}{2} \sigma_{k+1,N-k}} \right) e^{H_{k+1,N-k} \otimes \sigma_{k+1,N-k}} \quad (4.1)$$

where $k = 0, \dots, p$ and $\sigma_{ij} = \ln(1 + E_{i,j})$. This chain produces the deformation $\mathcal{F}_{\mathcal{B}_{p<0}} : U(\mathfrak{sl}(N)) \rightarrow U_{\mathcal{P}}(\mathfrak{sl}(N))$. The deformed Hopf algebra $U_{\mathcal{B}}(\mathfrak{sl}(N))$ admits additional twists. One of them, $\Phi_{\mathcal{J}_{p<0}}^{\mathcal{R}}$, a multidimensional analogue of $(\Phi_{\mathcal{R}})^{-1}$, must transform $U_{\mathcal{B}}(\mathfrak{sl}(N))$ into the quantized algebra of peripheric type $\Phi_{\mathcal{J}_{p<0}}^{\mathcal{R}} : U_{\mathcal{B}}(\mathfrak{sl}(N)) \rightarrow U_{\mathcal{B}\mathcal{P}_{p<0}}(\mathfrak{sl}(N))$. To start the construction of the Hopf algebra $U_{\mathcal{B}\mathcal{P}_{p<0}}(\mathfrak{sl}(N))$ we separate the peripheric factors in the Jordanian twists

$$\Phi_{\mathcal{J}_k} = e^{H_{k+1,N-k} \otimes \sigma_{k+1,N-k}} = e^{H_{k+1,N-k}^{\mathcal{R}} \otimes \sigma_{k+1,N-k}} e^{H_{k+1,N-k}^{\mathcal{P}} \otimes \sigma_{k+1,N-k}} = \Phi_{\mathcal{J}_k}^{\mathcal{R}} \Phi_{\mathcal{J}_k}^{\mathcal{P}}.$$

Here

$$H_{k+1,N-k}^{\mathcal{P}} = \frac{1}{N} \left(N E_{k+1,k+1} - \sum_{s=1}^N E_{s,s} \right) \quad (4.2)$$

$$H_{k+1,N-k}^{\mathcal{R}} = H_{k+1,N-k} - H_{k+1,N-k}^{\mathcal{P}}. \quad (4.3)$$

When the factor $\Phi_{\mathcal{J}_k}^{\mathcal{R}}$ is dragged to the end of the extended twist $\mathcal{F}_{\mathcal{B}_k}$, the extension factors will be changed:

$$\mathcal{F}_{\mathcal{B}_k} = \Phi_{\mathcal{J}_k}^{\mathcal{R}} \left(\prod_{s=k+2}^{N-k-1} e^{E_{k+1,s} \otimes E_{s,N-k}} \right) e^{H_{k+1,N-k}^{\mathcal{P}} \otimes \sigma_{k+1,N-k}} = \Phi_{\mathcal{J}_k}^{\mathcal{R}} \mathcal{F}_{\mathcal{B}_k}^{\mathcal{P}}. \quad (4.4)$$

One can check that $\mathcal{F}_{\mathcal{B}_k}^{\mathcal{P}}$ are the multidimensional peripheric extended twists. In particular, the Cartan element $H_{k+1,N-k}^{\mathcal{P}}$ commutes with all the second constituent generators $\{E_{s,N-k}; s = k+2, \dots, N-k-1\}$ (in formula (2.21) these are the elements $L_{\lambda_0^k - \lambda'}$). Applying $\mathcal{F}_{\mathcal{B}_k}$ to the primitive subalgebra $U(\mathfrak{sl}(N-2k))$ (this is just the situation that happens after the action of the k first factors $\mathcal{F}_{\mathcal{B}_{k-1}} \dots \mathcal{F}_{\mathcal{B}_0}$) it preserves the primitivity of all the first constituent generators $\{E_{k+1,s}; s = k+2, \dots, N-k-1\}$.

The factor $\Phi_{\mathcal{J}_k}^{\mathcal{R}}$ can be further pushed to the very end of the chain because it commutes with all the subsequent links $\{\mathcal{F}_{\mathcal{B}_s}; s > k\}$. The factors $\Phi_{\mathcal{J}_k}^{\mathcal{R}}$ with different indices k commute with each other. Performing the factorizations (4.4) in all the links and collecting the factors $\Phi_{\mathcal{J}_k}^{\mathcal{R}}$ at the end of the chain we get the following expression:

$$\mathcal{F}_{\mathcal{B}_{p<0}} \equiv \mathcal{F}_{\mathcal{B}_p} \mathcal{F}_{\mathcal{B}_{p-1}} \dots \mathcal{F}_{\mathcal{B}_0} = \Phi_{\mathcal{J}_p}^{\mathcal{R}} \Phi_{\mathcal{J}_{p-1}}^{\mathcal{R}} \dots \Phi_{\mathcal{J}_0}^{\mathcal{R}} \mathcal{F}_{\mathcal{B}_p}^{\mathcal{P}} \mathcal{F}_{\mathcal{B}_{p-1}}^{\mathcal{P}} \dots \mathcal{F}_{\mathcal{B}_0}^{\mathcal{P}} = \Phi_{\mathcal{J}_{p<0}}^{\mathcal{R}} \mathcal{F}_{\mathcal{B}_{p<0}}^{\mathcal{P}}. \quad (4.5)$$

We know [11] that $\mathcal{F}_{\mathcal{B}_{p<0}}$ is the twist for $U(\mathfrak{sl}(N))$. In the deformed algebra $U_{\mathcal{B}_{p<0}}(\mathfrak{sl}(N))$ we have $(p+1)$ primitive elements of the type $\sigma_{k+1,N-k}$. It can easily be checked that all the elements $H_{k+1,N-k}^{\mathcal{R}}$ are also primitive in $U_{\mathcal{B}_{p<0}}(\mathfrak{sl}(N))$. So, the integral factor $\Phi_{\mathcal{J}_{p<0}}^{\mathcal{R}}$ consists of Reshetikhin twists for commuting elements $\{H_{k+1,N-k}^{\mathcal{R}}, E_{k+1,N-k}; k = 0, \dots, p\}$. This means that $\Phi_{\mathcal{J}_{p<0}}^{\mathcal{R}}$ is a twist for $U_{\mathcal{B}_{p<0}}(\mathfrak{sl}(N))$, the same is true for $(\Phi_{\mathcal{J}_{p<0}}^{\mathcal{R}})^{-1}$. Consequently, the composition $(\Phi_{\mathcal{J}_{p<0}}^{\mathcal{R}})^{-1} \mathcal{F}_{\mathcal{B}_{p<0}}^{\mathcal{P}}$ is a twist for $U(\mathfrak{sl}(N))$ and it immediately follows from (4.5) that $\mathcal{F}_{\mathcal{B}_{p<0}}^{\mathcal{P}}$ is also a twist, $\mathcal{F}_{\mathcal{B}_{p<0}}^{\mathcal{P}} : U(\mathfrak{sl}(N)) \rightarrow U_{\mathcal{B}\mathcal{P}_{p<0}}(\mathfrak{sl}(N))$

$$\mathcal{F}_{\mathcal{B}_{p<0}}^{\mathcal{P}} = \prod_{k=0}^p \mathcal{F}_{\mathcal{B}_k}^{\mathcal{P}} = \prod_{k=0}^p \left(\prod_{s=k+2}^{N-k-1} e^{E_{k+1,s} \otimes E_{s,N-k}} \right) e^{H_{k+1,N-k}^{\mathcal{P}} \otimes \sigma_{k+1,N-k}}. \quad (4.6)$$

This composition of basic twisting factors is the necessary peripheric analogue of the chain of extended twists for $sl(N)$ and is called the peripheric chain of twists. The chain is full when the number of links (for $\mathbf{B}^+(sl(N))$) is maximal: $N/2$ for even and $(N+1)/2$ for odd N .

5. Peripheric chains as quantization tools

Let us consider $N = 2n$ for the even case and $N = 2n - 1$ for the odd one. The twisted algebra $U_{\mathcal{BP}_{p < 0}}(sl(N))$ has $z = (p+1)$ primitive elements $\sigma_{k+1, N-k}$ and $(N-1-z)$ primitive Cartan generators. Each link $\mathcal{F}_{B_k}^P$ of the chain $\mathcal{F}_{B_{p < 0}}^P$ is a peripheric extended twist. After applying it to $U_{\mathcal{BP}_{(k-1) < 0}}(sl(N))$ we get $(N-2k-3)$ primitive generators $L_{\lambda^{k'}}$ corresponding to the constituent roots $\lambda^{k'}$ (such that $\lambda^{k'} + \lambda^{k''} = \lambda_0^k$). In our case these are the elements $\{E_{k+1, s} \mid s = k+2, \dots, N-k-1\}$. The next link $\mathcal{F}_{B_{k+1}}^P$ preserves the primitivity only of one of them, $E_{k+1, N-k-1}$. Thus in $U_{\mathcal{BP}_{p < 0}}(sl(N))$ (after having applied z links) we get $(N-p-2)$ additional primitive generators (with respect to the effect of the canonical chain). The following statement will be useful for applications.

Lemma 5.1. *The ‘matreshka’ effect [11] is valid for peripheric chains.*

Proof. The peripheric link $\mathcal{F}_{B_k}^P$ is the function of the tensor invariant for the subalgebra $\mathfrak{g}_{\lambda_0^k}^\perp$ and being applied to $U_{\mathcal{BP}_{(k-1) < 0}}(\mathfrak{g}_{\lambda_0^k}^\perp)$ such twist cannot change its coproducts. On the first step of the inductive process (with $\mathcal{F}_{B_0}^P$) all the vectors in \mathfrak{g} have primitive coproducts. \square

In particular, the Hopf algebra $U_{\mathcal{BP}_{p < 0}}(sl(N))$ contains the subalgebra $U(sl(N-2z))$ whose generators are primitive.

In the following we shall consider the most important case: the full peripheric chains $\mathcal{F}_{B_{(N-n-1) < 0}}^P$ for the enveloping algebra $U(sl(N))$. They have $z = (N-n)$ links and the twisted algebras $U_{\mathcal{BP}_{(N-n-1) < 0}}(sl(N))$ contain $(N-n)$ primitive elements $\sigma_k \equiv \sigma_{k+1, N-k}$, $(n-1)$ primitive Cartan generators and $(n-1)$ additional primitive generators $E_k^P \equiv E_{k+1, N-k-1}$. This provides possibilities for enlarging the sequence of twists in the peripheric chain by the new twisting factors. As demonstrated in section 3 these possibilities are of two kinds: additional Reshetikhin and additional Jordanian basic twisting factors.

5.1. Additional Reshetikhin BTFs

Note that the primitive elements $I_m \in \{\sigma_k, E_l^P \mid k, l = 0, \dots, z-1\}$ commute. Consequently, the following factor:

$$\mathcal{F}_{\mathcal{R}} = \exp \{ \beta^{mn} I_m \otimes I_n \}$$

is the solution of the Drinfeld equations (2.1) for the algebra $U_{\mathcal{BP}_{(N-n-1) < 0}}(sl(N))$. The coefficients β^{mn} are arbitrary. Thus, the composition

$$\mathcal{F}_{\mathcal{RB}}^P = \mathcal{F}_{\mathcal{R}} \mathcal{F}_{B_{(N-n-1) < 0}}^P$$

is the twisting element for $U(sl(N))$,

$$\mathcal{F}_{\mathcal{RB}}^P : U(sl(N)) \rightarrow U_{\mathcal{RB}_{(N-n-1) < 0}}(sl(N)).$$

Each link $\mathcal{F}_{B_k}^P$ in $\mathcal{F}_{B_{(N-n-1) < 0}}^P$ can have an independent parameter ψ_{k+1} [11],

$$\mathcal{F}_{B_k}^P(\psi_{k+1}) = \left(\prod_{s=k+2}^{N-k-1} e^{\psi_{k+1} E_{k+1, s} \otimes E_{s, N-k}} \right) e^{H_{k+1, N-k}^P \otimes \sigma_{k+1, N-k}(\psi_{k+1})}. \quad (5.1)$$

Let all the variables of $\mathcal{F}_{\mathcal{RB}}^{\mathcal{P}}$ be proportional to the overall deformation parameter ξ , i.e. $\beta^{mn} \Rightarrow \xi \beta^{mn}$ and $\psi_{k+1} \Rightarrow \xi \psi_{k+1}$. The universal element (2.2)

$$\mathcal{R}_{\mathcal{RB}}^{\mathcal{P}}(\xi) = (\mathcal{F}_{\mathcal{RB}}^{\mathcal{P}}(\xi))_{21} (\mathcal{F}_{\mathcal{RB}}^{\mathcal{P}}(\xi))^{-1}$$

corresponds to the classical r -matrix

$$\begin{aligned} r_{\mathcal{RB}}^{\mathcal{P}} = & \sum_{k=0}^p \psi_{k+1} \left(H_{k+1,N-k}^{\mathcal{P}} \wedge E_{k+1,N-k} + \sum_{s=k+2}^{N-k-1} E_{k+1,s} \wedge E_{s,N-k} \right) \\ & + \sum_{n,m=1}^{2z} \beta^{mn} J_m \wedge J_n. \end{aligned} \quad (5.2)$$

Here $J_m \in \{E_{k+1,N-k}, E_l^{\mathcal{P}} \mid k, l = 0, \dots, z-1\}$. The difference between $r_{\mathcal{RB}}^{\mathcal{P}}$ and the r -matrix for the canonical chain

$$r_{\mathcal{B}} = \sum_{k=0}^p \psi_{k+1} \left(H_{k+1,N-k} \wedge E_{k+1,N-k} + \sum_{s=k+2}^{N-k-1} E_{k+1,s} \wedge E_{s,N-k} \right)$$

is essential. This is clearly seen when the corresponding Frobenius forms are compared

$$\omega_{\mathcal{RB}}^{\mathcal{P}} = \sum_{k=0}^p \chi_{k+1} E_{k+1,N-k}^*([,]) + \sum_{n,m=1}^{2z} \phi^{mn} K_m^* \wedge K_n^*$$

and

$$\omega_{\mathcal{B}} = \sum_{k=0}^p \chi_{k+1} E_{k+1,N-k}^*([,])$$

where $K_m \in \{H_{k+1,N-k}^{\mathcal{P}}, E_l^{\mathcal{P}} \mid k, l = 0, \dots, z-1\}$. The coefficients χ_{k+1} and ϕ^{mn} are proportional to ψ_{k+1} and β^{mn} , respectively. Note that $\omega_{\mathcal{RB}}^{\mathcal{P}} = \omega_{\mathcal{B}} + \sum_{n,m=1}^{2z} \phi^{mn} K_m^* \wedge K_n^*$ and when all ϕ^{mn} are equal to zero the forms coincide despite the fact that the corresponding terms in the r -matrix contain different Cartan generators. Moreover, as we had already mentioned in section 4, the carrier algebras for $r_{\mathcal{B}}$ and $r_{\mathcal{RB}}^{\mathcal{P}}(\phi^{mn} = 0)$ are different. In the general case (with nonzero ϕ^{mn}) the ω -forms are also different: $\omega_{\mathcal{B}}$ is a coboundary while $\omega_{\mathcal{RB}}^{\mathcal{P}}$ is cohomologically nontrivial.

The result is that the peripheric chains provide the possibility of quantizing explicitly the new class of classical r -matrices (5.2), defined on carrier subalgebras of the $\mathbf{L}^{\mathcal{P}}$ -type, whose ω -forms are cohomologically nontrivial.

5.2. Additional Jordanian BTFs

Let us consider the peripheric chain $\mathcal{F}_{\mathcal{B}(N-n-1) \prec 0}^{\mathcal{P}}$ and its possible extensions related to the primitive Cartan generators $\{H_i^{\perp}\}$ and the additional primitive elements $\{E_i^{\mathcal{P}}\} (i = 1, \dots, n-1)$ belonging to the twisted Hopf algebra $U_{\mathcal{BP}(N-n-1) \prec 0}(sl(N))$. Here, we consider only the generators $E_i^{\mathcal{P}}$ because the alternative pairs $\{H_i^{\perp}, \sigma_k\}$ appear also in the canonical case and were treated earlier (see for example [18]). The dual elements $\{H_i^{\perp*}\}$ generate in the root space the hyperplane orthogonal to the initial roots $\{\lambda_0^k \mid k = 0, \dots, z-1\}$ of the chain $\mathcal{F}_{\mathcal{B}(N-n-1) \prec 0}^{\mathcal{P}}$. On this hyperplane we fix the set of basic elements $\{H_i^{\perp}\}$ and consider it together with the set $\{E_i^{\mathcal{P}}\}$:

$$H_i^{\perp} = \frac{N-2i}{N} \sum_{l=1}^N E_{l,l} - \frac{2i}{N} \sum_{m=i+1}^{N-i} E_{m,m} \quad E_j^{\mathcal{P}} = E_{j,N-j} \quad (5.3)$$

with $i, j = 1, \dots, n-1$. On the joint space generated by $\{H_i^\perp, E_j^{\mathcal{P}}\}$ we have a Lie algebra with the nonzero commutators,

$$[H_i^\perp, E_j^{\mathcal{P}}] = \delta_{ij} E_j^{\mathcal{P}}.$$

In other words, this is the direct sum of $(n-1)$ 2D Borel subalgebras \mathbf{B}_i . This guarantees the possibility of enlarging the twisting element $\mathcal{F}_{\mathcal{B}(N-n-1)<0}^{\mathcal{P}}$ with the additional (independent) Jordanian basic factors $\Phi_{J_i} = e^{H_i^\perp \otimes \sigma_i}$.

Let $\mathcal{F}_{\mathcal{JB}}^{\mathcal{P}}$ be the maximal enlargement of the full peripheric chain with $(n-1)$ additional JBFs,

$$\mathcal{F}_{\mathcal{JB}}^{\mathcal{P}} = \left(\prod_{i=1}^{n-1} e^{H_i^\perp \otimes \sigma_i} \right) \mathcal{F}_{\mathcal{B}(N-n-1)<0}^{\mathcal{P}}.$$

Formally, the carrier subalgebra of this chain coincides with the Borel subalgebra $\mathbf{B}^+(sl(N))$. It contains all the raising operators $\{E_{lm} \mid l, m = 1, \dots, N; l < m\}$, $(N-n)$ generators $H_{k,N-k}^{\mathcal{P}}$ and $(n-1)$ generators H_i^\perp . However, as we shall demonstrate below, the dimension of the carrier for the classical r -matrix $r_{\mathcal{JB}}^{\mathcal{P}}$ remains the same as in the full chain $\mathcal{F}_{\mathcal{B}(N-n-1)<0}^{\mathcal{P}}$.

To study the properties of the enlarged chain in detail we must introduce in $\mathcal{F}_{\mathcal{JB}}^{\mathcal{P}}$ the full set of parameters. This can be done by means of an involutive automorphism of the carrier subalgebra and the discrete transformations of subalgebras inside the carrier. It is sufficient to consider the maximal nilpotent subalgebra $\mathbf{N}^+(sl(N)) \subset \mathbf{B}^+(sl(N))$. We perform the smooth scaling of the generators in $\mathbf{N}^+(sl(N))$:

$$\{E_{lm} \Rightarrow \alpha_{lm} E_{lm} \mid \alpha_{lm} \in \mathbb{C}; l, m = 1, \dots, N; l < m\}.$$

The scaling factors α_{lm} are as follows:

$$\begin{aligned} \text{for } \left\{ E_{lm} \mid \begin{array}{l} l = 1, \dots, z-1 \\ m = 2, \dots, z \\ l < m \end{array} \right\} & \quad \alpha_{lm} = \frac{\psi_l}{\psi_m} \\ \text{for } \left\{ E_{lm} \mid \begin{array}{l} l = 1, \dots, z \\ m = z+1, \dots, N \end{array} \right\} & \quad \alpha_{lm} = \psi_l \zeta_{(N-m)} \quad \zeta_0 = 1 \\ \text{for } \left\{ E_{lm} \mid \begin{array}{l} l = z+1, \dots, N-1 \\ m = z+2, \dots, N \\ l < m \end{array} \right\} & \quad \alpha_{lm} = \frac{\zeta_{(N-m)}}{\zeta_{(N-l)}}. \end{aligned} \quad (5.4)$$

All the parameters $\{\psi_l, \zeta_i \mid l = 1, \dots, z; i = 1, \dots, N-z-1\}$ are independent. The first subset $\{\psi_l\}$ consists of the parameters of links, already introduced in $\mathcal{F}_{\mathcal{B}_k}^{\mathcal{P}}(\psi_{k+1})$ (see (5.1)), the entries of the second subset $\{\zeta_i\}$ refer to the Jordanian factors $\Phi_{J_i} = e^{H_i^\perp \otimes \sigma_i}$. Notice that each argument $E_i^{\mathcal{P}}$ of σ_i is scaled by a product of parameters, $E_i^{\mathcal{P}} \Rightarrow (\psi_i \zeta_j) E_i^{\mathcal{P}}$, because it is already scaled in the corresponding link $\mathcal{F}_{\mathcal{B}_{i-1}}^{\mathcal{P}}(\psi_i)$ of the chain. The discrete parameters describe the property: in any link of the chain the extension factor can be switched off. Only the full extension

$$\Phi_{\mathcal{E}_k}(\psi_{k+1} \kappa_{k+1}) = \prod_{\lambda' \in \pi'_k} \Phi_{\mathcal{E}_{\lambda'}}(\psi_{k+1} \kappa_{k+1}) = \prod_{s=k+2}^{N-k-1} e^{\psi_{k+1} \kappa_{k+1} E_{k+1,s} \otimes E_{s,N-k}}$$

has the character of the basic extending factor (2.11) with the continuous parameter ψ_{k+1} and the discrete parameter $\kappa_{k+1} = 0, 1$. The separate EFs inside $\Phi_{\mathcal{E}_k}$ conserve their independence and can be switched off. But, in the general case, such cancellation ruins the matreshka

effect and the structure of the chain is, hence, lost. The parametrized enlarged chain has the form

$$\begin{aligned}\mathcal{F}_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_l, \kappa_l, \zeta_i\}) &= \left(\prod_{i=1}^{n-1} e^{H_i^{\perp} \otimes \sigma_i(\psi_i \kappa_i \zeta_i)} \right) \mathcal{F}_{\mathcal{B}(N-n-1) < 0}^{\mathcal{P}}(\{\psi_l, \kappa_l, \zeta_i\}) \\ &= \left(\prod_{i=1}^{n-1} e^{H_i^{\perp} \otimes \sigma_i(\psi_i \kappa_i \zeta_i)} \right) \prod_{k=0}^z \left(\prod_{s=k+2}^{N-k-1} e^{\psi_{k+1} \zeta_k \kappa_{k+1} E_{k+1,s} \otimes E_{s,N-k}} \right) e^{H_{k+1,N-k}^{\mathcal{P}} \otimes \sigma_{k+1,N-k}(\psi_{k+1} \zeta_k)}.\end{aligned}\quad (5.5)$$

Any number of links $\mathcal{F}_{\mathcal{B}_{i-1}}^{\mathcal{P}}(\{\psi_l, \kappa_l, \zeta_i\})$ and JBFs $\Phi_{J_i}(\psi_i \kappa_i \zeta_i) = e^{H_i^{\perp} \otimes \sigma_i(\psi_i \kappa_i \zeta_i)}$ can be switched off in the enlarged chain. This is ensured by the following rearrangement of parameters:

$$\begin{aligned}\psi_l &= v_l \prod_{r=1}^{l-1} \frac{v_r}{\rho_r} & \zeta_i &= \prod_{r=1}^i \frac{\rho_r}{v_r} \\ \mathcal{F}_{\mathcal{JB}}^{\mathcal{P}}(\{v_l, \kappa_l, \rho_i\}) &= \left(\prod_{i=1}^{n-1} e^{H_i^{\perp} \otimes \sigma_i(\kappa_i \rho_i)} \right) \mathcal{F}_{\mathcal{B}(N-n-1) < 0}^{\mathcal{P}}(\{v_l, \kappa_l\}) \\ &= \left(\prod_{i=1}^{n-1} e^{H_i^{\perp} \otimes \sigma_i(\kappa_i \rho_i)} \right) \prod_{k=0}^z \left(\prod_{s=k+2}^{N-k-1} e^{v_{k+1} \kappa_{k+1} E_{k+1,s} \otimes E_{s,N-k}} \right) e^{H_{k+1,N-k}^{\mathcal{P}} \otimes \sigma_{k+1,N-k}(v_{k+1})}.\end{aligned}\quad (5.6)$$

From the structural point of view the independence of the additional factors $\Phi_{J_i}(\kappa_i \rho_i)$ is obvious. Due to the matreshka effect switching off a link $\mathcal{F}_{\mathcal{B}(N-n-1) < 0}^{\mathcal{P}}(\{v_l, \kappa_l\})$ in $\mathcal{F}_{\mathcal{JB}}^{\mathcal{P}}(\{v_l, \rho_i, \kappa_l\})$ cannot prevent any of $E_i^{\mathcal{P}}$ being primitive in $U_{\mathcal{B}(N-n-1) < 0}^{\mathcal{P}}(sl(N))$. When an extension $\Phi_{\mathcal{E}_l}(v_l \kappa_l)$ is switched off by putting $\kappa_l = 0$ the corresponding factor $e^{H_l^{\perp} \otimes \sigma_l(\kappa_l \rho_l)}$ also vanishes. This correlates with the fact that when the extension $\Phi_{\mathcal{E}_l}(v_l \kappa_l)$ is absent (the corresponding link contains only a JF) the coproduct $\Delta_{\mathcal{J}}(E_l^{\mathcal{P}})$ is not primitive and the additional JF Φ_{J_l} has to be cancelled. Strictly speaking, even in such a situation the number of additional factors Φ_{J_i} can be preserved. Instead of $\Delta_{\mathcal{J}}(E_l^{\mathcal{P}})$ we have the primitive $\Delta_{\mathcal{J}}(E_{N-l,N-l+1})$ and the factor $e^{H_l^{\perp} \otimes \sigma_l(\kappa_l \rho_l)}$ can be substituted by $e^{H_l^{\perp} \otimes \sigma_{N-l,N-l+1}(\rho_l)}$.

When the extensions are switched off in all the links the initial chain degenerates into the twist

$$\mathcal{F}_{\mathcal{B}}^{\mathcal{P}}(\{v_l, 0, \rho_i\}) = \prod_{k=0}^z e^{H_{k+1,N-k}^{\mathcal{P}} \otimes \sigma_{k+1,N-k}(v_{k+1})}$$

(with the special ‘peripheric’ choice of the Cartan generators $H_{k+1,N-k}^{\mathcal{P}}$). This multi-Jordanian twist can be enlarged by $(n-1)$ JFs,

$$\mathcal{F}_{\mathcal{JB}}^{\mathcal{P}}(\{v_l, 0, \rho_i\}) = \left(\prod_{i=1}^{n-1} e^{H_i^{\perp} \otimes \sigma_{N-l,N-l+1}(\rho_i)} \right) \prod_{k=0}^z e^{H_{k+1,N-k}^{\mathcal{P}} \otimes \sigma_{k+1,N-k}(v_{k+1})}.$$

Now we shall consider the quasi-classical limit for the enlarged chain $\mathcal{F}_{\mathcal{JB}}^{\mathcal{P}}(\{v_l, \kappa_l, \rho_i\})$. To simplify the formulae we put $\{\kappa_l = 1 \mid l = 1, \dots, n-1\}$. The quasi-classical limit for $\mathcal{F}_{\mathcal{JB}}^{\mathcal{P}}(\{v_l, 1, \rho_i\}) \equiv \mathcal{F}_{\mathcal{JB}}^{\mathcal{P}}(\{v_l, \rho_i\})$ is obtained through the substitution $\psi_k \Rightarrow \xi \psi_k$ in (5.5) or $v_l \Rightarrow \xi v_l$, $\rho_i \Rightarrow \xi \rho_i$ in (5.6) with the overall deformation parameter ξ . In the neighbourhood of the origin the \mathcal{R} -matrix $\mathcal{R}_{\mathcal{JB}}^{\mathcal{P}}$ has the expansion

$$\begin{aligned}\mathcal{R}_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_l, \zeta_i; \xi\}) &= 1 \otimes 1 + \xi r_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_l, \zeta_i\}) + \mathcal{O}(\xi) \\ &= 1 \otimes 1 + \xi r_{\mathcal{JB}}^{\mathcal{P}}(\{v_l, \rho_i\}) + \mathcal{O}(\xi).\end{aligned}$$

This means that the deformation $U(sl(N)) \rightarrow U_{\mathcal{F}_{\mathcal{B}(N-n-1)}^{\mathcal{P}}}(sl(N))$ performed by the twist $\mathcal{F}_{\mathcal{B}}^{\mathcal{P}}(\{\psi_l, \zeta_i; \xi\})$ can be treated as a quantization of the classical mechanical system described by the r -matrix

$$\begin{aligned} r_{\mathcal{F}_{\mathcal{B}}}^{\mathcal{P}} &= \sum_{k=0}^p \psi_{k+1} \zeta_k \left(H_{k+1, N-k}^{\mathcal{P}} \wedge E_{k+1, N-k} + \sum_{s=k+2}^{N-k-1} E_{k+1, s} \wedge E_{s, N-k} \right) + \sum_{i=1}^{n-1} \psi_i \zeta_i H_i^{\perp} \wedge E_{i, N-i} \\ &= \sum_{k=0}^p v_{k+1} \left(H_{k+1, N-k}^{\mathcal{P}} \wedge E_{k+1, N-k} + \sum_{s=k+2}^{N-k-1} E_{k+1, s} \wedge E_{s, N-k} \right) + \sum_{i=1}^{n-1} \rho_i H_i^{\perp} \wedge E_{i, N-i}. \end{aligned} \quad (5.7)$$

It can be easily checked that on the space of $\mathbf{B}^+(sl(N))$ this r -matrix is degenerate.

Now we shall show that the carrier subalgebra of $r_{\mathcal{F}_{\mathcal{B}}}^{\mathcal{P}}$ has the dimension $(N^2 + N - 2n)/2$ (the same as in the case of $r_{\mathcal{B}}^{\mathcal{P}}$).

Lemma 5.2. *The carrier algebra of the classical r -matrix $r_{\mathcal{F}_{\mathcal{B}}}^{\mathcal{P}}(\{\psi_l, \zeta_i\})$ (5.7) is equivalent to the $((N^2 + N - 2n)/2)$ -dimensional subalgebra of $\mathbf{B}^+(sl(N))$ generated by the following sets of elements:*

$$\begin{aligned} H_{i, N-i+1}^{\mathcal{P}} & \quad i = 1, \dots, z \\ H_j^{\perp} & \quad j = 1, \dots, n-1 \\ E_{lm} & \quad \left\{ \begin{array}{lll} l = 1, \dots, z-1 & m = 2, \dots, z & l < m \\ l = 1, \dots, z & m = z+1, \dots, N & \\ l = z+1, \dots, N-2 & m = z+3, \dots, N & l < m \end{array} \right\}. \end{aligned} \quad (5.8)$$

Proof. It is obvious that set (5.8) generates a subalgebra, that we denote $\mathfrak{g}_{\mathcal{F}_{\mathcal{B}}}^{\mathcal{P}}$. It contains the Cartan subalgebra of $sl(N)$ and all the positive roots operators except those corresponding to the following subset of basic roots:

$$\{\lambda_{z+i} = e_{z+i} - e_{z+i+1} \mid i = 1, \dots, N - z - 1\}.$$

Let us construct the smooth set of injections φ of $\mathfrak{g}_{\mathcal{F}_{\mathcal{B}}}^{\mathcal{P}}$ into $\mathbf{B}^+(sl(N))$ depending on the parameters $\{\zeta_i\}$:

$$\varphi(\{\zeta_i\}) : \begin{cases} H_{i, N-i+1}^{\mathcal{P}} \rightarrow H_{i, N-i+1}^{\mathcal{P}} & i = 1, \dots, z \\ H_j^{\perp} \rightarrow -B_j(\{\zeta_i\}) & j = 1, \dots, n-1 \\ E_{lm} \rightarrow E_{lm} & l = 1, \dots, z-1 \quad m = 2, \dots, z \quad l < m \\ E_{lm} \rightarrow A_{lm}(\{\zeta_i\}) & l = 1, \dots, z \quad m = z+1, \dots, N \\ E_{lm} \rightarrow C_{lm}(\{\zeta_i\}) & l = z+1, \dots, N-2 \quad m = z+3, \dots, N \quad l < m \end{cases} \quad (5.9)$$

where

$$\begin{aligned} B_j(\{\zeta_i\}) &= \sum_{s=1}^j \frac{\zeta_{j-s}}{\zeta_j} E_{N-j, N-j+s} - H_j^{\perp} \\ A_{lm}(\{\zeta_i\}) &= \frac{1}{\zeta_{N-m}} \sum_{s=0}^{N-m} \zeta_{N-m-s} E_{l, m+s} \\ C_{lm}(\{\zeta_i\}) &= E_{lm} - \frac{\zeta_{N-l}}{\zeta_{N-m}} \left(\sum_{s=m}^N \frac{\zeta_{N-s}}{\zeta_{N-l+1}} E_{l-1, s} + \sum_{s=1}^{N-m} \frac{\zeta_{N-m-s}}{\zeta_{N-l}} E_{l, m+s} \right). \end{aligned} \quad (5.10)$$

Direct computations show that the map $\varphi(\{\zeta_i\})$ is an automorphism of $\mathfrak{g}_{\mathcal{JB}}^{\mathcal{P}}$. Let us denote by $\{D_s\}$ the basis of $\mathfrak{g}_{\mathcal{JB}}^{\mathcal{P}}$ obtained as the image of the natural basis (5.8)

$$\varphi(\{\zeta_i\}) : \{H_{i,N-i+1}^{\mathcal{P}}, H_j^{\perp}, E_{pt}\} \rightarrow \{D_s\} \equiv \{H_{i,N-i+1}^{\mathcal{P}}, B_j(\{\zeta_i\}), E_{lm}, A_{lm}(\{\zeta_i\}), C_{lm}(\{\zeta_i\})\}.$$

The r -matrix $r_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_l, \zeta_i\})$ can be expressed in terms of the generators $\{D_s\}$. First let us rewrite it as follows:

$$\begin{aligned} r_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_l, \zeta_i\}) &= \sum_{k=0}^p \psi_{k+1} \zeta_k E_{k+1, N-k-1} \wedge \left(E_{N-k-1, N-k} - \frac{\zeta_{k+1}}{\zeta_k} H_{j+1}^{\perp} \right) \\ &+ \sum_{k=0}^p \psi_{k+1} \zeta_k \left(H_{k+1, N-k}^{\mathcal{P}} \wedge E_{k+1, N-k} + \sum_{s=k+2}^{N-k-2} E_{k+1, s} \wedge E_{s, N-k} \right). \end{aligned} \quad (5.11)$$

Consider the restriction of the map φ_E (5.9) to the subspace generated by

$$\begin{array}{lll} E_{lm} & \text{for } l = 1, \dots, z-1 & m = 2, \dots, z \quad l < m \\ E_{lm} & \text{for } l = 1, \dots, z & m = z+1, \dots, N \\ E_{lm} & \text{for } l = z+1, \dots, N-2 & m = z+3, \dots, N \quad l < m. \end{array}$$

Formulae (5.10) describe the decompositions $\varphi_E(E_{ij}) = D_{s(ij)} = (\varphi_E)_s^{pt} E_{pt}$. It is easy to see that the matrix $\{(\varphi_E)_s^{pt}\}$ is invertible and $E_{pt} = (\varphi_E^{-1})_{pt}^s D_s$ are the linear combinations of the elements E_{ij} and A_{ij} whose indices are $\{i \geq l, j \leq m\}$. The remaining entries of $r_{\mathcal{JB}}^{\mathcal{P}}$ (5.11) are the combinations $(C_{N-k-1, N-k} - \frac{\zeta_{k+1}}{\zeta_k} H_{j+1}^{\perp})$. According to definition (5.10) these terms depend only on B_j and the generators $\{\bar{C}_{pt} \mid p = z+1, \dots, N-2; t = z+3, \dots, N; p < t\}$. As a result all the arguments of the r -matrix $r_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_l, \zeta_i\})$ are proved to belong to the space generated by $H_{i, N-i+1}^{\mathcal{P}}, B_j, E_{lm}, A_{lm}$ and C_{lm} , that is the algebra $\mathfrak{g}_{\mathcal{JB}}^{\mathcal{P}}$ described above:

$$r_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_l, \zeta_i\}) \in \varphi(\mathfrak{g}_{\mathcal{JB}}^{\mathcal{P}}) \wedge \varphi(\mathfrak{g}_{\mathcal{JB}}^{\mathcal{P}}) \approx \mathfrak{g}_{\mathcal{JB}}^{\mathcal{P}} \wedge \mathfrak{g}_{\mathcal{JB}}^{\mathcal{P}}.$$

In this presentation the $r_{\mathcal{JB}}^{\mathcal{P}}$ -matrix is nondegenerate. This proves the lemma. \square

We can calculate the symplectic form $\omega_{\mathcal{JB}}^{\mathcal{P}}$ corresponding to $r_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_l, \zeta_i\})$. It is a special kind of coboundary generated by the basic forms $A_{l,m}^*([,])$ with $\{l = 1, \dots, z; m = z+1, \dots, N-l+1\}$. As far as the map φ is an isomorphism we can use the canonical basis (5.8). In this case the corresponding coboundaries are $E_{l,m}^*([,])$. The symplectic form $\omega_{\mathcal{JB}}^{\mathcal{P}}$ has a very simple structure:

$$\omega_{\mathcal{JB}}^{\mathcal{P}}(\{\zeta_i\}) = - \sum_{l=1}^z \sum_{m=z+1}^{N-l+1} \frac{1}{\alpha_{lm}} E_{l,m}^*([,]) \quad (5.12)$$

where $\alpha_{lm} = \alpha_{lm}(\{\zeta_i\})$ are the scaling factors defined in (5.4).

6. Examples

We shall demonstrate the properties of the enlarged peripheric chains presenting explicitly the expressions corresponding to three special cases: $U(sl(4))$, $U(sl(7))$ and $U(sl(3))$.

6.1. $U(sl(4))$

This is the simplest case where the peripheric properties can be visualized because four is the lowest dimension of the space whose algebra of linear transformations has the nontrivial chain

of extended twists [11] characterized by the indices $j = 1, i = 1, 2; n = 2; z = N - n = 2$,

$$\begin{aligned}\mathcal{F}_{\mathcal{B}_{1 \leftarrow 0}}^{\mathcal{P}} &= \prod_{k=0}^1 \mathcal{F}_{\mathcal{B}_k}^{\mathcal{P}} = \prod_{k=0}^1 \left(\prod_{s=k+2}^{3-k} e^{E_{k+1,s} \otimes E_{s,4-k}} \right) e^{H_{k+1,4-k}^{\mathcal{P}} \otimes \sigma_{k+1,4-k}} \\ &= e^{H_{2,3}^{\mathcal{P}} \otimes \sigma_{2,3}} e^{E_{1,2} \otimes E_{2,4}} e^{E_{1,3} \otimes E_{3,4}} e^{H_{1,4}^{\mathcal{P}} \otimes \sigma_{1,4}}.\end{aligned}\quad (6.1)$$

Note that the extension factor in the second link is trivial. This chain differs from the canonical one by the Cartan generators in the Jordanian factors:

$$H_{1,4}^{\mathcal{P}} = \frac{1}{4} \left(3E_{1,1} - \sum_{s=2}^4 E_{s,s} \right) \quad H_{2,3}^{\mathcal{P}} = \frac{1}{4} (-E_{1,1} + 3E_{2,2} - E_{3,3} - E_{4,4}).$$

The basic vector $(H_1^\perp)^*$ of the hyperplane orthogonal to $\lambda_{1,4} = e_1 - e_4$ and $\lambda_{2,3} = e_2 - e_3$ corresponds to the generator

$$H_1^\perp = \frac{1}{2} (E_{1,1} - E_{2,2} - E_{3,3} + E_{4,4}).$$

The coproducts of the generators of $\mathbf{B}^+(sl(4))$ can be considered as a modification of the coproducts $\Delta_{\mathcal{B}_{1 \leftarrow 0}}$ (twisted by a canonical chain) due to the Reshetikhin ‘rotation’ of the type $(\Phi_{\mathcal{J}_{1 \leftarrow 0}}^{\mathcal{R}})^{-1}$ (see expression (4.5)):

$$\begin{aligned}\Delta_{\mathcal{B}_{1 \leftarrow 0}}^{\mathcal{P}}(E_{1,2}) &= E_{1,2} \otimes e^{-\sigma_{1,4}} + 1 \otimes E_{1,2} - H_{2,3}^{\mathcal{P}} \otimes E_{13} e^{-\sigma_{2,3}} \\ \Delta_{\mathcal{B}_{1 \leftarrow 0}}^{\mathcal{P}}(E_{1,3}) &= E_{1,3} \otimes 1 + 1 \otimes E_{1,3} \\ \Delta_{\mathcal{B}_{1 \leftarrow 0}}^{\mathcal{P}}(E_{1,4}) &= E_{1,4} \otimes e^{\sigma_{1,4}} + 1 \otimes E_{1,4} \\ \Delta_{\mathcal{B}_{1 \leftarrow 0}}^{\mathcal{P}}(E_{2,3}) &= E_{2,3} \otimes e^{\sigma_{2,3}} + 1 \otimes E_{2,3} \\ \Delta_{\mathcal{B}_{1 \leftarrow 0}}^{\mathcal{P}}(E_{2,4}) &= E_{2,4} \otimes e^{\sigma_{2,3}} + e^{\sigma_{1,4}} \otimes E_{2,4} \\ \Delta_{\mathcal{B}_{1 \leftarrow 0}}^{\mathcal{P}}(E_{3,4}) &= E_{3,4} \otimes 1 + e^{\sigma_{1,4}} \otimes E_{3,4} + H_{2,3}^{\mathcal{P}} \otimes E_{24} e^{-\sigma_{2,3}} \\ \Delta_{\mathcal{B}_{1 \leftarrow 0}}^{\mathcal{P}}(H_{1,4}^{\mathcal{P}}) &= H_{1,4}^{\mathcal{P}} \otimes e^{-\sigma_{1,4}} + 1 \otimes H_{1,4}^{\mathcal{P}} - E_{1,3} \otimes E_{3,4} e^{-\sigma_{1,4}} \\ &\quad - (E_{1,2} + H_{2,3}^{\mathcal{P}} E_{1,3}) \otimes E_{2,4} e^{-\sigma_{1,4} - \sigma_{2,3}} \\ \Delta_{\mathcal{B}_{1 \leftarrow 0}}^{\mathcal{P}}(H_{2,3}^{\mathcal{P}}) &= H_{2,3}^{\mathcal{P}} \otimes e^{-\sigma_{2,3}} + 1 \otimes H_{2,3}^{\mathcal{P}} \\ \Delta_{\mathcal{B}_{1 \leftarrow 0}}^{\mathcal{P}}(H_1^\perp) &= H_1^\perp \otimes 1 + 1 \otimes H_1^\perp.\end{aligned}\quad (6.2)$$

We have here two additional primitive elements

$$E_1^{\mathcal{P}} = E_{1,3} \quad H_1^\perp$$

which determine the Borel algebra

$$[H_1^\perp, E_1^{\mathcal{P}}] = E_1^{\mathcal{P}}.$$

The twist $\mathcal{F}_{\mathcal{B}_{1 \leftarrow 0}}^{\mathcal{P}}$ (6.1) can be enlarged by the factor $\Phi_{J_1} = e^{H_1^\perp \otimes \sigma_1}$, with $\sigma_1 = \ln(1 + E_1^{\mathcal{P}})$,

$$\begin{aligned}\mathcal{F}_{\mathcal{J}\mathcal{B}}^{\mathcal{P}} &= e^{H_1^\perp \otimes \sigma_1} \mathcal{F}_{\mathcal{B}_{1 \leftarrow 0}}^{\mathcal{P}} \\ &= e^{H_1^\perp \otimes \sigma_1} e^{H_{2,3}^{\mathcal{P}} \otimes \sigma_{2,3}} e^{E_{1,2} \otimes E_{2,4}} e^{E_{1,3} \otimes E_{3,4}} e^{H_{1,4}^{\mathcal{P}} \otimes \sigma_{1,4}}.\end{aligned}\quad (6.3)$$

The enlarged twist corresponds to the \mathcal{R} -matrix

$$\mathcal{R}_{\mathcal{J}\mathcal{B}}^{\mathcal{P}} = e^{\sigma_1 \otimes H_1^\perp} (\mathcal{F}_{\mathcal{B}_{1 \leftarrow 0}}^{\mathcal{P}})_{21} (\mathcal{F}_{\mathcal{B}_{1 \leftarrow 0}}^{\mathcal{P}})^{-1} e^{-H_1^\perp \otimes \sigma_1}.$$

This universal element (supplied with the full set of deformation parameters $\{\psi_l \Rightarrow \xi \psi_l, \varsigma_1 \mid l = 1, 2\}$ with the overall parameter ξ) can be considered as a quantization of the classical r -matrix $r_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_l, \varsigma_i\})$,

$$r_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_l, \varsigma_1\}) = \psi_1 E_{1,3} \wedge (E_{3,4} - \varsigma_1 H_1^\perp) + \psi_2 \varsigma_1 H_{2,3}^{\mathcal{P}} \wedge E_{2,3} \\ + \psi_1 (H_{1,4}^{\mathcal{P}} \wedge E_{1,4} + E_{1,2} \wedge E_{2,4}). \quad (6.4)$$

The Lie–Poisson structure fixed by $r_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_l, \varsigma_i\})$ can be redefined in terms of the algebra $\mathfrak{g}_{\mathcal{JB}}^{\mathcal{P}}$ that in our case is 8D generated by $\{H_{1,4}^{\mathcal{P}}, H_{2,3}^{\mathcal{P}}, H_1^\perp, E_{p,t} \mid p = 1, 2; t = 2, 3, 4; p < t\}$. This is an algebra of motion over the 4D space with the translations $\mathfrak{g}_{\mathcal{P}} = \{E_{p,t} \mid p = 1, 2; t = 3, 4\}$ and the subalgebra $\mathfrak{g}_{\mathcal{H}}$ containing the operator $E_{1,2}$ and the Cartan generators $\{H_{1,4}^{\mathcal{P}}, H_{2,3}^{\mathcal{P}}, H_1^\perp\}$,

$$\mathfrak{g}_{\mathcal{JB}}^{\mathcal{P}} = \mathfrak{g}_{\mathcal{H}} \vdash \mathfrak{g}_{\mathcal{P}}.$$

In terms of the image $\varphi(\mathfrak{g}_{\mathcal{JB}}^{\mathcal{P}})$ generated by

$$B_1(\{\varsigma_1\}) = \frac{1}{\varsigma_1} E_{3,4} - H_1^\perp \quad E_{12} \\ A_{13}(\varsigma_1) = E_{1,3} + \frac{1}{\varsigma_1} E_{1,4} \quad A_{14} = E_{14} \quad H_{1,4}^{\mathcal{P}} \\ A_{23}(\varsigma_1) = E_{2,3} + \frac{1}{\varsigma_1} E_{2,4} \quad A_{24} = E_{24} \quad H_{2,3}^{\mathcal{P}}$$

this r -matrix looks like

$$r_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_l, \varsigma_1\}) = \psi_1 \varsigma_1 \left(A_{13} - \frac{1}{\varsigma_1} A_{14} \right) \wedge B_1 + \psi_2 \varsigma_1 H_{2,3}^{\mathcal{P}} \wedge \left(A_{23} - \frac{1}{\varsigma_1} A_{24} \right) \\ + \psi_1 (H_{1,4}^{\mathcal{P}} \wedge A_{1,4} + E_{1,2} \wedge A_{2,4}). \quad (6.5)$$

The corresponding symplectic form (in terms of the algebra $\mathfrak{g}_{\mathcal{JB}}^{\mathcal{P}}$)

$$\omega_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_i, \varsigma_1\}) = - \sum_{l=1}^2 \sum_{m=3}^{5-l} \frac{1}{\alpha_{lm}} E_{l,m}^*([\cdot, \cdot]) \\ = - \left(\frac{1}{\psi_1 \varsigma_1} E_{1,3}^* + \frac{1}{\psi_1} E_{1,4}^* + \frac{1}{\psi_2 \varsigma_1} E_{2,3}^* \right) ([\cdot, \cdot])$$

defines $\mathfrak{g}_{\mathcal{JB}}^{\mathcal{P}}$ as a Frobenius algebra.

Expression (6.3) for the enlarged peripheric twist shows that the Hopf algebra $U_B^{\mathcal{P}}(sl(4))$ can be twisted further by the JF $e^{H_1^\perp \otimes \sigma_1}$:

$$U(sl(4)) \xrightarrow{\mathcal{F}_{B_1 \rightarrow 0}^{\mathcal{P}}} U_B^{\mathcal{P}}(sl(4)) \xrightarrow{e^{H_1^\perp \otimes \sigma_1}} U_{\mathcal{JB}}^{\mathcal{P}}(sl(4)).$$

The final co-structure $\Delta_{\mathcal{JB}}^{\mathcal{P}}$ can be obtained as $\Delta_{\mathcal{JB}}^{\mathcal{P}} = e^{\text{ad}(H_1^\perp \otimes \sigma_1)} \circ \Delta_B^{\mathcal{P}}$ and is defined by the set of coproducts:

$$\Delta_{\mathcal{JB}}^{\mathcal{P}}(E_{1,2}) = E_{1,2} \otimes e^{\sigma_{1,3} - \sigma_{1,4}} + 1 \otimes E_{1,2} - H_{2,3}^{\mathcal{P}} \otimes E_{13} e^{-\sigma_{2,3}} \\ \Delta_{\mathcal{JB}}^{\mathcal{P}}(E_{1,3}) = E_{1,3} \otimes e^{\sigma_{1,3}} + 1 \otimes E_{1,3} \\ \Delta_{\mathcal{JB}}^{\mathcal{P}}(E_{1,4}) = E_{1,4} \otimes e^{\sigma_{1,4}} + 1 \otimes E_{1,4} \\ \Delta_{\mathcal{JB}}^{\mathcal{P}}(E_{2,3}) = E_{2,3} \otimes e^{\sigma_{2,3}} + 1 \otimes E_{2,3} \\ \Delta_{\mathcal{JB}}^{\mathcal{P}}(E_{2,4}) = E_{2,4} \otimes e^{\sigma_{2,3} - \sigma_{1,3}} + e^{\sigma_{1,4}} \otimes E_{2,4} \\ \Delta_{\mathcal{JB}}^{\mathcal{P}}(E_{3,4}) = E_{3,4} \otimes e^{-\sigma_{1,3}} + e^{\sigma_{1,4}} \otimes E_{3,4} + H_1^\perp e^{\sigma_{1,4}} \otimes (e^{\sigma_{1,4}} - 1) e^{-\sigma_{1,3}} + H_{2,3}^{\mathcal{P}} \otimes E_{24} e^{-\sigma_{2,3}} \\ \Delta_{\mathcal{JB}}^{\mathcal{P}}(H_{1,4}^{\mathcal{P}}) = H_{1,4}^{\mathcal{P}} \otimes e^{-\sigma_{1,4}} + 1 \otimes H_{1,4}^{\mathcal{P}} - H_1^\perp \otimes (e^{-\sigma_{1,3}} - 1) - (e^{\sigma_{1,3}} - 1) \otimes E_{3,4} e^{-\sigma_{1,4} + \sigma_{1,3}} \\ - H_1^\perp (e^{\sigma_{1,3}} - 1) \otimes (1 - e^{-\sigma_{1,4}}) - (E_{1,2} + H_{2,3}^{\mathcal{P}} E_{1,3}) \otimes E_{2,4} e^{-\sigma_{1,4} - \sigma_{2,3} + \sigma_{1,3}} \\ \Delta_{\mathcal{JB}}^{\mathcal{P}}(H_{2,3}^{\mathcal{P}}) = H_{2,3}^{\mathcal{P}} \otimes e^{-\sigma_{2,3}} + 1 \otimes H_{2,3}^{\mathcal{P}} \\ \Delta_{\mathcal{JB}}^{\mathcal{P}}(H_1^\perp) = H_1^\perp \otimes e^{-\sigma_{1,3}} + 1 \otimes H_1^\perp.$$

All the Cartan generators of $sl(4)$ participate in the deformation that leads to this Hopf algebra $U_{\mathcal{F}_B}^{\mathcal{P}}(sl(4))$.

In the ordinary injection of the Poincaré algebra in $sl(4)$ (in the corresponding real form) the Cartan generators are identified with the diagonalizable rotation (usually l_{12}), boost (n_{03}) and the dilatation operator. Note that here the commutative subalgebra of generators $\{A_{ij} = E_{ij} \mid i = 1, 2; j = 3, 4\}$ is just the subalgebra of Poincaré translations. All of them have the especially simple quasi-primitive coproducts. The reason is that in the twist $\mathcal{F}_{\mathcal{F}_B}^{\mathcal{P}}$ the number of Jordanian factors $\mathcal{F}_{\mathcal{J}}$ is maximal and each of them is attached to one of the ‘momenta’.

6.2. $U(sl(7))$

Now we shall briefly expose the case of dimension seven to show some peculiarities of odd dimension $N = 2n - 1$. We also want the algebra $U(sl(N))$ to be sufficiently large to contain nontrivially the generators of the type $C_{lm}(\{\zeta_i\})$.

In this case the full peripheric chain has three links ($n = 4; i, j = 1, 2, 3; z = N - n = 3$)

$$\begin{aligned} \mathcal{F}_{B_{2 \leftarrow 0}}^{\mathcal{P}} &= \prod_{k=0}^{\leftarrow 2} \mathcal{F}_{B_k}^{\mathcal{P}} = \prod_{k=0}^{\leftarrow 2} \left(\prod_{s=k+2}^{6-k} e^{E_{k+1,s} \otimes E_{s,7-k}} \right) e^{H_{k+1,7-k}^{\mathcal{P}} \otimes \sigma_{k+1,7-k}} \\ &= e^{E_{3,4} \otimes E_{4,5}} e^{H_{3,5}^{\mathcal{P}} \otimes \sigma_{3,5}} e^{E_{2,3} \otimes E_{3,6}} e^{E_{2,4} \otimes E_{4,6}} e^{E_{2,5} \otimes E_{5,6}} e^{H_{2,6}^{\mathcal{P}} \otimes \sigma_{2,6}} \\ &\quad \times e^{E_{1,2} \otimes E_{2,7}} e^{E_{1,3} \otimes E_{3,7}} e^{E_{1,4} \otimes E_{4,7}} e^{E_{1,5} \otimes E_{5,7}} e^{E_{1,6} \otimes E_{6,7}} e^{H_{1,7}^{\mathcal{P}} \otimes \sigma_{1,7}}. \end{aligned} \quad (6.6)$$

The Cartan generators are fixed according to the prescription (4.2):

$$H_{k+1,N-k}^{\mathcal{P}} = \frac{1}{N} \left(N E_{k+1,k+1} - \sum_{s=1}^N E_{s,s} \right) \quad k = 0, 1, 2.$$

The dimension of the subalgebra \mathbf{H}^{\perp} in $\mathbf{H}(sl(7))$ that remains primitive after the twisting performed by the peripheric chain $\mathcal{F}_{B_{2 \leftarrow 0}}^{\mathcal{P}}$

$$\mathcal{F}_{B_{2 \leftarrow 0}}^{\mathcal{P}} : U(sl(7)) \rightarrow U_B^{\mathcal{P}}(sl(7))$$

coincides with the dimension of $\mathbf{H}^{\mathcal{P}}$. We choose the following three basic elements (see (5.3)):

$$H_i^{\perp} = \frac{N - 2i}{N} \sum_{l=1}^N E_{l,l} - \sum_{m=i+1}^{N-i} E_{m,m} \quad i = 1, 2, 3. \quad (6.7)$$

It is easy to check that the subalgebras $\mathbf{H}^{\mathcal{P}}$ and \mathbf{H}^{\perp} are primitive in the twisted algebra $U_B^{\mathcal{P}}(sl(N))$:

$$\begin{aligned} \Delta_{B_{2 \leftarrow 0}}^{\mathcal{P}}(H_i^{\perp}) &= H_i^{\perp} \otimes 1 + 1 \otimes H_i^{\perp} \\ \Delta_{B_{2 \leftarrow 0}}^{\mathcal{P}}(E_j^{\mathcal{P}}) &= E_j^{\mathcal{P}} \otimes 1 + 1 \otimes E_j^{\mathcal{P}} \end{aligned} \quad i, j = 1, 2, 3$$

where $E_j^{\mathcal{P}} = E_{j,N-j}$. The generators $\{H_i^{\perp}, E_j^{\mathcal{P}}\}$ form three (mutually commuting) Borel subalgebras:

$$[H_i^{\perp}, E_j^{\mathcal{P}}] = \delta_{ij} E_j^{\mathcal{P}}.$$

This indicates that the peripheric chain $\mathcal{F}_{B_{2 \leftarrow 0}}^{\mathcal{P}}$ (6.6) can be enlarged by the factors $\Phi_{J_i} = e^{H_i^{\perp} \otimes \sigma_i}$, with $\sigma_i = \ln(1 + E_i^{\mathcal{P}})$:

$$\mathcal{F}_{\mathcal{F}_B}^{\mathcal{P}} = \left(\prod_{i=1}^3 e^{H_i^{\perp} \otimes \sigma_i} \right) \mathcal{F}_{B_{1 \leftarrow 0}}^{\mathcal{P}}.$$

The corresponding universal \mathcal{R} -matrix has the form

$$\mathcal{R}_{\mathcal{JB}}^{\mathcal{P}} = \left(\prod_{i=1}^3 e^{\sigma_i \otimes H_i^\perp} \right) (\mathcal{F}_{\mathcal{B}_{2,0}}^{\mathcal{P}})_{21} (\mathcal{F}_{\mathcal{B}_{2,0}}^{\mathcal{P}})^{-1} \left(\prod_{i=1}^3 e^{-H_i^\perp \otimes \sigma_i} \right).$$

Note that the additional JFs commute.

We have six deformation parameters: $\{\psi_l \Rightarrow \xi \psi_l, \varsigma_l \mid l = 1, 2, 3\}$. The expansion of $\mathcal{R}_{\mathcal{JB}}^{\mathcal{P}}(\psi_l, \varsigma_l; \xi)$ with respect to ξ exhibits the following classical r -matrix:

$$\begin{aligned} r_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_l, \varsigma_l\}) = & \psi_1 \left[H_1^{\mathcal{P}} \wedge E_{1,7} + \sum_{k=2}^5 E_{1,k} \wedge E_{k,7} + E_{1,6} \wedge (E_{6,7} - \varsigma_1 H_1^\perp) \right] \\ & + \psi_2 \varsigma_1 \left[H_2^{\mathcal{P}} \wedge E_{2,6} + \sum_{k=3}^4 E_{2,k} \wedge E_{k,6} + E_{2,5} \wedge \left(E_{5,6} - \frac{\varsigma_2}{\varsigma_1} H_2^\perp \right) \right] \\ & + \psi_3 \varsigma_2 \left[H_3^{\mathcal{P}} \wedge E_{3,5} + E_{3,4} \wedge \left(E_{4,5} - \frac{\varsigma_3}{\varsigma_2} H_3^\perp \right) \right]. \end{aligned} \quad (6.8)$$

The carrier algebra $\mathfrak{g}_{\mathcal{JB}}^{\mathcal{P}}$ of the r -matrix $r_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_l, \varsigma_l\})$ is 24D with the generators $\{H_i^{\mathcal{P}}, H_i^\perp, \{E_{p,t} \mid p = 1, 2, 3; t = 2, \dots, 7; p < t\}, \{E_{4,6}, E_{4,7}, E_{5,7}\}\}$. This is an algebra of motion over the 12D space with translations $\mathfrak{g}_{\mathcal{P}} = \{E_{p,t} \mid p = 1, 2, 3; t = 4, 5, 6, 7\}$ and the subalgebra $\mathfrak{g}_{\mathcal{H}}$ that is a direct sum of two 6D algebras $\mathfrak{g}_{\mathcal{H}} = \mathfrak{g}'_{\mathcal{H}} \oplus \mathfrak{g}''_{\mathcal{H}}$. Each direct summand contains three Cartan and three positive root generators: $\mathfrak{g}'_{\mathcal{H}} = \{H_i^{\mathcal{P}}, E_{p,t} \mid p, t = 1, 2, 3; p < t\}$ and $\mathfrak{g}''_{\mathcal{H}} = \{H_i^\perp, E_{4,6}; E_{4,7}, E_{5,7}\}$. In these terms $\mathfrak{g}_{\mathcal{JB}}^{\mathcal{P}}$ is a semi-direct sum,

$$\mathfrak{g}_{\mathcal{JB}}^{\mathcal{P}} = (\mathfrak{g}'_{\mathcal{H}} \oplus \mathfrak{g}''_{\mathcal{H}}) \ltimes \mathfrak{g}_{\mathcal{P}}. \quad (6.9)$$

The image $\varphi(\mathfrak{g}_{\mathcal{JB}}^{\mathcal{P}})$ of the map φ contains the generators

$$\begin{aligned} B_1(\{\varsigma_1\}) &= \frac{1}{\varsigma_1} E_{6,7} - H_1^\perp & E_{i,j} \quad i, j = 1, 2, 3 \\ B_2(\{\varsigma_{1,2}\}) &= \frac{1}{\varsigma_2} E_{5,6} + \frac{1}{\varsigma_1 \varsigma_2} E_{5,7} - H_2^\perp & H_i^{\mathcal{P}} \\ B_2(\{\varsigma_{1,2}\}) &= \frac{1}{\varsigma_3} E_{4,5} + \frac{1}{\varsigma_2 \varsigma_3} E_{4,6} + \frac{1}{\varsigma_1 \varsigma_2 \varsigma_3} E_{4,7} - H_3^\perp & E_{i,7} \\ A_{i,4} &= E_{i,4} + \frac{1}{\varsigma_3} E_{i,5} + \frac{1}{\varsigma_2 \varsigma_3} E_{i,6} + \frac{1}{\varsigma_1 \varsigma_2 \varsigma_3} E_{i,7} & C_{4,6} = E_{4,6} + \frac{1}{\varsigma_1} E_{4,7} \\ A_{i,5} &= E_{i,5} + \frac{1}{\varsigma_2} E_{i,6} + \frac{1}{\varsigma_1 \varsigma_2} E_{i,7} & C_{4,7} = E_{4,6} \\ A_{i,6} &= E_{i,6} + \frac{1}{\varsigma_1} E_{i,7} & C_{5,7} = E_{5,7} - \frac{\varsigma_2}{\varsigma_3} E_{4,7}. \end{aligned}$$

In these terms the r -matrix $r_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_l, \varsigma_l\})$ has the form

$$\begin{aligned} r_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_l, \varsigma_l\}) = & \psi_1 \left[H_1^{\mathcal{P}} \wedge A_{1,7} + E_{1,2} \wedge A_{2,7} + E_{1,3} \wedge A_{3,7} + \left(A_{14} - \frac{\varsigma_2}{\varsigma_3} A_{15} \right) \wedge A_{47} \right. \\ & + \left(A_{15} - \frac{\varsigma_1}{\varsigma_2} A_{16} \right) \wedge \left(A_{57} + \frac{\varsigma_2}{\varsigma_3} A_{47} \right) + \left(A_{16} - \frac{1}{\varsigma_1} A_{17} \right) \wedge \varsigma_1 B_1 \Big] \\ & + \varsigma_1 \psi_2 \left[H_2^{\mathcal{P}} \wedge \left(A_{2,6} - \frac{1}{\varsigma_1} A_{27} \right) + E_{2,3} \wedge \left(A_{36} - \frac{1}{\varsigma_1} A_{37} \right) \right. \\ & + \left. \left(A_{24} - \frac{\varsigma_2}{\varsigma_3} A_{25} \right) \wedge \left(A_{46} - \frac{1}{\varsigma_1} A_{47} \right) \right] \end{aligned}$$

$$\begin{aligned}
& + \left(A_{25} - \frac{\varsigma_1}{\varsigma_2} A_{26} \right) \wedge \left(\frac{\varsigma_2}{\varsigma_1} B_2 - \frac{1}{\varsigma_1} \left(A_{57} + \frac{\varsigma_2}{\varsigma_3} A_{47} \right) \right) \Big] \\
& + \varsigma_2 \psi_3 \left[H_3^{\mathcal{P}} \wedge \left(A_{3,5} - \frac{\varsigma_1}{\varsigma_2} A_{36} \right) + \left(A_{3,4} - \frac{\varsigma_2}{\varsigma_3} A_{35} \right) \wedge \left(\frac{\varsigma_3}{\varsigma_2} B_3 - \frac{\varsigma_1}{\varsigma_2} A_{46} \right) \right].
\end{aligned} \tag{6.10}$$

This expression enables us to calculate the corresponding form ω that has a simple structure in terms of the algebra $\mathfrak{g}_{\mathcal{JB}}^{\mathcal{P}}$:

$$\omega_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_l, \varsigma_l\}) = - \sum_{l=1}^3 \sum_{m=4}^{8-l} \frac{1}{\alpha_{lm}} E_{l,m}^*([,]) \quad \alpha_{lm} = \psi_l \varsigma_{(N-m)}.$$

Thus, when the Hopf algebra $U_{\mathcal{B}_{2<0}}(sl(7))$ is twisted by the additional JFs $\prod_{i=1}^3 e^{H_i^{\perp} \otimes \sigma_i}$, the result

$$\prod_{i=1}^3 e^{H_i^{\perp} \otimes \sigma_i} : U_{\mathcal{B}_{2<0}}(sl(7)) \rightarrow U_{\mathcal{JB}}^{\mathcal{P}}(sl(7))$$

is a quantization of the initial Hopf algebra $U(sl(7))$ in the direction defined by the r -matrix $r_{\mathcal{JB}}^{\mathcal{P}}(\{\psi_l, \varsigma_l\})$ ((6.8) or (6.10)) with carrier algebra $\mathfrak{g}_{\mathcal{JB}}^{\mathcal{P}}$ (see (6.9)).

6.3. $U(sl(3))$

To conclude the examples the degenerate case of the enlarged peripheric chain is worth mentioning, the PET $\mathcal{F}_{\mathcal{E}}^{\mathcal{P}} = e^{E_{1,2} \otimes E_{2,3}} e^{H^{\mathcal{P}} \otimes \sigma_{1,3}}$ in $U(sl(3))$ equipped with the additional Jordanian factor:

$$\mathcal{F}_{\mathcal{JE}}^{\mathcal{P}} = e^{H_{13}^{\perp} \otimes \sigma_{12}(\varsigma)} e^{\psi E_{1,2} \otimes E_{2,3}} e^{H^{\mathcal{P}} \otimes \sigma_{1,3}(\psi)} \tag{6.11}$$

(see (3.2)). Here the generators $H^{\mathcal{P}}$ and $H_{13}^{\perp} = H^{\perp}$ were defined in section 3, formula (3.1). The corresponding classical r -matrix (3.3) is evidently one-parametric:

$$r_{\mathcal{JE}}^{\mathcal{P}}(\varsigma) = H^{\mathcal{P}} \wedge E_{13} + E_{12} \wedge (E_{23} - \varsigma H_{13}^{\perp}). \tag{6.12}$$

Its carrier algebra $\mathfrak{g}_{\mathcal{JE}}^{\mathcal{P}}$ is 4D with two Cartan and two ‘translation’ generators: $\{H^{\mathcal{P}}, H_{13}^{\perp}, E_{12}, E_{13}\}$. The natural basis in the image $\varphi(\mathfrak{g}_{\mathcal{JE}}^{\mathcal{P}})$ of the map φ is formed by the elements:

$$\begin{aligned}
B(\{\varsigma\}) &= \frac{1}{\varsigma} E_{23} - H_{13}^{\perp} & H^{\mathcal{P}} \\
A_{12} &= E_{12} + \frac{1}{\varsigma} E_{13} & E_{13}.
\end{aligned}$$

The ω -form corresponding to the r -matrix (6.12) is

$$\omega_{\mathcal{JE}}^{\mathcal{P}}(\varsigma) = -\frac{1}{\varsigma} E_{12}^*([,]) - E_{13}^*([,]).$$

The important fact here is that for the 4D carrier algebra $\mathfrak{g}_{\mathcal{JE}}^{\mathcal{P}}$ we can easily compose the other twisting element that contains no external generators. Consider the basic generators $\{H_{12}^{\perp}, H_{13}^{\perp}, E_{12}, E_{13}\}$ correlated with the direct sum structure $\mathfrak{g}_{\mathcal{JE}}^{\mathcal{P}} = \mathbf{B}_2(H_{12}^{\perp}, E_{13}) \oplus \mathbf{B}_2(H_{13}^{\perp}, E_{12})$. This is obviously the two-Jordanian situation and the corresponding twist is

$$\mathcal{F}_{\mathcal{JJ}} = e^{H_{13}^{\perp} \otimes \sigma_{12}(\varsigma)} e^{H_{12}^{\perp} \otimes \sigma_{13}(\psi)}.$$

Note that the r -matrix

$$r_{\mathcal{JJ}}(\varsigma) = H_{12}^{\perp} \wedge E_{13} + \eta H_{13}^{\perp} \wedge E_{12}$$

is equivalent to (6.12). In this particular case we can construct two different twists, $\mathcal{F}_{\mathcal{J}\mathcal{J}}$ and $\mathcal{F}_{\mathcal{J}\mathcal{E}}^{\mathcal{P}}$, that quantize the r -matrix $r_{\mathcal{J}\mathcal{J}} \approx r_{\mathcal{J}\mathcal{E}}^{\mathcal{P}}$. One of them has the carrier algebra $\mathfrak{g}_{\mathcal{J}\mathcal{E}}^{\mathcal{P}}$, the other depends on the full set of generators of $\mathcal{B}^+(sl(3))$. In other words, the twist $\mathcal{F}_{\mathcal{J}\mathcal{E}}^{\mathcal{P}}$ uses not only the adjoint operators of the elements of $\mathfrak{g}_{\mathcal{J}\mathcal{E}}^{\mathcal{P}}$ but also the morphisms external with respect to $\mathfrak{g}_{\mathcal{J}\mathcal{E}}^{\mathcal{P}}$.

This is the explicit demonstration of the fact that there exist different quantizations of one and the same Lie–Poisson structure.

7. Conclusions

We have explicitly quantized the sets of Lie–Poisson structures defined on the groups of motion. When they are of the $r_{\mathcal{J}\mathcal{B}}^{\mathcal{P}}$ -type the carrier algebras of the r -matrix and the twist $\mathcal{F}_{\mathcal{J}\mathcal{B}}^{\mathcal{P}}(\{v_l, \kappa_l, \rho_i\})$ are different ($\mathfrak{g}_{\mathcal{J}\mathcal{B}}^{\mathcal{P}}$ and $\mathbf{B}^+(sl(N))$, respectively). It is necessary to stress that the defining space of the twisting element $\mathcal{F}_{\mathcal{J}\mathcal{B}}^{\mathcal{P}}(\{v_l, \kappa_l, \rho_i\})$ cannot be restricted to the subspace $U(\mathfrak{g}_{\mathcal{J}\mathcal{B}}^{\mathcal{P}}) \otimes U(\mathfrak{g}_{\mathcal{J}\mathcal{B}}^{\mathcal{P}})$. The quantization of the r -matrix $r_{\mathcal{J}\mathcal{B}}^{\mathcal{P}}$ with carrier subalgebra $\mathfrak{g}_{\mathcal{J}\mathcal{B}}^{\mathcal{P}}$ was performed by the twist $\mathcal{F}_{\mathcal{J}\mathcal{B}}^{\mathcal{P}}(\{v_l, \kappa_l, \rho_i\})$ with carrier $\mathbf{B}^+(sl(N))$. On the other hand, it is well known [6] that to find the twisting element it is sufficient to know the classical r -matrix, there exists the solution of the Drinfeld equations corresponding to r and defined over $\mathfrak{g}_{\mathcal{J}\mathcal{B}}^{\mathcal{P}}$. It was demonstrated above that this is not the only way to perform the explicit twisting. The alternative possibility is to inject the carrier into the larger algebra and to search for the solutions there. Thus, to perform the quantization corresponding to the carrier $\mathfrak{g}_{\mathcal{J}\mathcal{B}}^{\mathcal{P}}$ we have used the enlarged peripheric twist $\mathcal{F}_{\mathcal{J}\mathcal{B}}^{\mathcal{P}}(\{v_l, \kappa_l, \rho_i\})$ that includes the adjoint operators external with respect to $\mathfrak{g}_{\mathcal{J}\mathcal{B}}^{\mathcal{P}}$. This method can be used to facilitate the solution of the Drinfeld equations: in the nonordinary cases it might be reasonable to enlarge the algebra and thus include the external morphisms.

In the general situation the explicit form of the twisting element with the carrier $\mathfrak{g}_{\mathcal{J}\mathcal{B}}^{\mathcal{P}}$ (that depends only on the generators of $\mathfrak{g}_{\mathcal{J}\mathcal{B}}^{\mathcal{P}}$) is unknown to us. The only exception is the degenerate case of $U(sl(3))$ considered at the end of the previous section. The same properties can be found when N is arbitrary but the chain is degenerate and consists of only one link. In this single link the extension can contain $N - 2$ basic extending factors and (when the whole link is peripheric) can lead to $N - 2$ additional primitive generators. The enlarged peripheric twist will have $N - 2$ additional Jordanian factors. At the same time here the carrier $\mathfrak{g}_{\mathcal{J}\mathcal{E}}^{\mathcal{P}}$ is obviously equivalent to the direct sum of $N - 1$ commuting Borel subalgebras and the corresponding multi-Jordanian twist can be composed.

The algebras $\mathfrak{g}_{\mathcal{J}\mathcal{B}}^{\mathcal{P}} = \mathfrak{g}_{\mathcal{H}} \vdash \mathfrak{g}_{\mathcal{P}}$ are algebras of motion of special type. They have the $(z(N - z))$ -dimensional subspace corresponding to the translations $A_{lm} \in \mathfrak{g}_{\mathcal{P}}$. The subalgebra $\mathfrak{g}_{\mathcal{H}}$ contains the Borel subalgebra $\mathbf{B}^+(gl(z))$, which is the multidimensional analogue of the dilatation operator in the conformal algebra, and the subalgebra $\mathbf{C}^+(gl(n - 1))$ that can be treated as a contraction of $\mathbf{B}^+(gl(n - 1))$. The Cartan subalgebra in $\mathfrak{g}_{\mathcal{H}}$ contains the ordinary set of simultaneously diagonalizable operators in the algebra of motion $\mathfrak{g}_{\mathcal{J}\mathcal{B}}^{\mathcal{P}}$ including the subset of dilatation-like operators B_i . The applications of the enlarged peripheric chains of twists studied in this paper to quantize Lie–Poisson structures defined on groups of motion could be of considerable interest.

To study the peripheric chains for simple Lie algebras of series A_n we have used the auxiliary Reshetikhin ‘rotation’ in the root space. This method is quite general and can be applied to an arbitrary simple algebra.

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